Arithmetic Progressions in Permutations

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Abstract: In this thesis we will examine progressions in permutations. We will analyze regular and circular permutations with progressions either modular or regular with rise one or two. We will calculate the number of permutations on the numbers 0 through $n-1$ containing x progressions. This is based largely on work by Riordan [1] and Dymàček and Lambert [2].

1 Introduction

We seek to answer how many permutations have x progressions. Given a permutation $\pi = (a_0, a_1, \ldots, a_{n-1})$, a progression of rise r, distance d, and length l in π is a sequence $(a_i, a_{i+d}, \ldots, a_{i+(l-1)d})$ for which

$$a_{i+jd} - a_{i+(j-1)d} = r$$

for $1 \leq j \leq l-1$. Hence, a progression is a sequence from our permutation for which both the terms and the indices of the terms have a fixed difference (not necessarily the same difference) between successive elements. If the equality above is instead an equivalence modulo n, we call the progression modular. We also consider progressions in circular permutations, in which case the indices in our progression are considered modulo n. We denote by $S_d^r(n, x, l)$ the set of all permutations on 0 through $n-1$ with x progressions of rise r, distance d, and length l. If we consider modular progressions we use $\overline{S}$ while C and $\overline{C}$ refer to circular permutations. We use a lower case s or c to denote the cardinality of the given set, that is $s_d^r(n, x, l)$ is the number of permutations on 0 through $n-1$ having x progressions of rise r, distance d, and length l. For the majority of this paper, we will be considering progressions of length 3. Thus it will be common to omit the argument l from our functions with the understanding that we consider l to be 3. If D and R are sets of integers, then $s_D^R(n, x)$ denotes the number of permutations on 0 through $n-1$ having a total of x progressions which have a distance in D and a rise in R.

2 History

Many authors have examined this problem in different forms. Most notably and relevant to this work are Riordan [1] and Dymàček and Lambert [2]. Riordan calculated the number of permutations on 0 through $n-1$ containing x progressions of rise and distance 1 with length 3. In our notation we have.

Theorem 1 (Riordan) for $n \geq 3$

$$s_1^1(n,x) = \delta_{x,0} n! + \sum_{k=1}^{n} (-1)^{k+x} \binom{k}{x} \sum_{i=0}^{k-1} \binom{k-1}{i} \binom{n-k-1}{k-1} (n-2k+i)!$$

where $\delta_{x,0}$ is the Kronecker delta. Dymàček and Lambert calculated $s_1^{\pm 1}(n,0)$, $\overline{s}_1^{\pm 1}(n,0)$, $c_1^{\pm 1}(n,0)$, and $\overline{c}_1^{\pm 1}(n,0)$ where ± 1 represents the set $\{1,-1\}$.

3 Examples

The permutation $(7,1,0,2,3,4,6,5)$ has three progressions: $(2,3,4)$ with $r = d = 1$, $(0,3,6)$ with $r = 3$ and $d = 2$, and $(1,3,5)$ with $r = 2$ and $d = 3$.

The permutation $(5,4,3,6,7,0,2,1)$ has two progressions: $(6,7,0)$ and $(5,4,3)$. These are both modular progressions.

The circular permutation $(0,2,1,3,5,4)$ has a total of 14 progressions. Some of these are $(2,3,4)$ and $(4,3,2)$ which are reverses of each other. Also $(4,0,2)$ and $(2,0,4)$ are progressions which are each other's reverse. Our definition also allows for the progressions $(0,3,0)$, $(3,0,3)$, and $(1,4,1)$ among others.

4 General Results

In this section we will develop some general results for relating various types of permutations. First there is a very tight relation between straight permutations with modular progressions and circular permutations with regular progressions. For α and β permutations we write $\alpha \equiv \beta$ and read α is modularly equal to β provided there exists an integer k such that $k + \alpha = \beta$. Clearly this is an equivalence relation on permutations.

Let $\alpha = (a_0, a_1, \ldots, a_{n-1})$ and $\alpha^{-1} = \beta = (b_0, b_1, \ldots, b_{n-1})$ where $b_{a_j} = j$. Then for $k \in \mathbb{Z}$, $\alpha + k = (a_0 + k, a_1 + k, \ldots, a_{n-1} + k)$ and $(\alpha + k)^{-1} = \gamma = (c_0, c_1, \ldots, c_{n-1})$ with $c_{a_j + k} = j$. We see then that γ and β are the same circular permutation since they differ only by shifting the index by k. Then the taking of inverses is a n-to-1 mapping from straight permutations to circular permutations where $\alpha \equiv \beta$ if and only if $\alpha^{-1} = \beta^{-1}$.

Theorem 2 For sets of integers D and R,

$$\overline{s}_D^R(n,x,l) = n \cdot c_R^D(n,x,l)$$

Proof Given a permutation π, $\big(a_i, a_{i+d}, \ldots, a_{i+(l-1)d}\big)$ is a modular progression of rise r, distance d, and length l if and only if $a_{i+jd} = ai + (j-1)d + r$ for $1 \le j \le l$ if and only if $b_{a_i+jd} = i + jd$ for $0 \le j \le l$ in π^{-1} if and only if $\big(b_{a_i}, b_{a_i+r}, \ldots, b_{a_i+(l-1)d}\big)$ is a progression of rise d, distance r, and length l in the circular permutation π^{-1}. ■

Note all equals above are equivalences modulo n, though if we take the equality to be regular we get a process that switches the rise and distance of regular progressions in straight permutations or modular progressions in circular permutations. Thus we have some easy corollaries.

Corollary 3 For sets of integers D and R,

$$
\begin{aligned}
s_D^R(n,x,l) &= s_R^D(n,x,l)\\
\overline{c}_D^R(n,x,l) &= \overline{c}_R^D(n,x,l)
\end{aligned}
$$

The following is another nice general result, but again we need a bit of preparation. Suppose that k is relatively prime to n and $\big(a_i, a_{i+d}, \ldots, a_{i+(l-1)d}\big)$ is a modular progression of rise r, distance d, and length l in a permutation π of length n. Then $a_{i+jd} - a_{i+(j-1)d} \equiv r \mod n$ for $0 < j < l$ if and only if $ka_{i+jd} - ka_{i+(j-1)d} \equiv kr \mod n$ for $0 < j < l$, that is, if and only if $\big(ka_i, ka_{i+d}, \ldots, ka_{i+(l-1)d}\big)$ is a progression of rise kr, distance d, and length l in the permutation $k\pi$. Then $\overline{s}_d^r(n,x,l) = \overline{s}_d^{kr}(n,x,l)$ with the same being true for $\overline{c}$. Applying the previous theorem we also get $c_d^r(n,x,l) = c_{kd}^r(n,x,l)$ with the same being true for $\overline{c}$.

Theorem 4 For r_1 and r_2 such that $\gcd(r_1,n) = \gcd(r_2,n)$ and d_1 and d_2 such that $\gcd(d_1,n) = \gcd(d_2,n)$ we have the following:

$$
\begin{aligned}
\overline{s}_d^{r_1}(n,x,l) &= \overline{s}_d^{r_2}(n,x,l)\\
\overline{c}_d^{r_1}(n,x,l) &= \overline{c}_d^{r_2}(n,x,l)\\
c_{d_1}^{r}(n,x,l) &= c_{d_2}^{r}(n,x,l)\\
\overline{c}_{d_1}^{r}(n,x,l) &= \overline{c}_{d_2}^{r}(n,x,l)
\end{aligned}
$$

Proof There exists some k relatively prime to n such that $kr_1 \equiv r_2 \mod n$. So by the above theorem $\overline{s}_d^{r_1}(n,x,l) = \overline{s}_d^{r_2}(n,x,l)$ with the same being true for $\overline{c}$. Similarly we see the last two equalities hold as well. ■

These previous two theorems in combination allow us to relate a large number of the quantities in which we are interested to each other.

5 Two Cases: $r = d = 1$ and $r = 2$, $d = 1$

In this section we will develop formulas for $s_1^2(n,x)$, $\overline{s}_1^1(n,x)$, $\overline{s}_1^2(n,x)$, $c_1^1(n,x)$, $c_1^2(n,x)$, $\overline{c}_1^1(n,x)$, and $\overline{c}_1^2(n,x)$. In all these cases we are considering progressions to be of length three. Note also that even though we say we have two cases, we really have seven things to calculate. Recall that $s_1^1(n,x)$ was calculated by Riordan.

Our technique of proof is similar to Riordan, but we formulate it differently. Recall that for $j > x$,

$$\begin{aligned}
\sum_{k=x}^{j}(-1)^{k-x}\binom{j}{k}\binom{k}{x} &= \sum_{k=x}^{j}(-1)^{k-x}\binom{j}{x}\binom{j-x}{k-x} \\
&= \binom{j}{x}\sum_{k=x}^{j}(-1)^{k-x}\binom{j-x}{k-x} \\
&= \binom{j}{x}\sum_{k=0}^{j-x}(-1)^{k}\binom{j-x}{k} = 0.
\end{aligned}$$

Using this we have,

$$\begin{aligned}
\sum_{k=0}^{n-4}(-1)^{k+x}\binom{k}{x}\sum_{j=k}^{n-4}\binom{j}{k}s_1^2(n,j) &= \sum_{k=x}^{n-4}(-1)^{k+x}\binom{k}{x}\sum_{j=k}^{n-4}\binom{j}{k}s_1^2(n,j) \\
&= \sum_{k=x}^{n-4}\sum_{j=k}^{n-4}(-1)^{k+x}\binom{k}{x}\binom{j}{k}s_1^2(n,j) \\
&= \sum_{j=x}^{n-4}s_1^2(n,j)\sum_{k=x}^{j}(-1)^{k-x}\binom{j}{k}\binom{k}{x} \\
&= s_1^2(n,x).
\end{aligned}$$

Therefore, our goal is to find an expression for

$$\sum_{j=k}^{n-4}\binom{j}{k}s_1^2(n,j)$$

that we can compute easily. Note that this is the number of permutations having at least k progressions (a straight permutation can have at most

$n-4$ progressions of rise 2 and distance 1). We can also replace $s_1^2(n,j)$ with $\overline{s}_1^2(n,j)$ or $c_1^2(n,j)$. If we adjust the upper limit on the sum to $n-2$ we can swap in $c_1^1(n,j)$, and if the upper limit were n we could swap in $\overline{s}_1^1(n,j)$, $\overline{c}_1^1(n,j)$, or $\overline{c}_1^2(n,j)$.

Focusing on $s_1^2(n,x)$, the number of permutations that contain some progression is $(n-2)!$ since we can permute the elements of the progression as one term. The number of permutations containing some two progressions is either $(n-3)!$ or $(n-4)!$ depending on whether the progressions fit together nicely like 024 and 246, or if they fit together poorly or not at like 024 and 468 or 024 and 135. For three progressions the our number of permutations is either $(n-4)!$, $(n-5)!$, or $(n-6)!$ corresponding to the initial elements of the progressions forming two, one, or no successions of rise 2 (a succession of rise k is adjacent pair of the form $(i, i+k)$). In general for k progressions the number of permutations containing them is $(n-2k+i)!$ where i is the number of successions in the initial elements when taken in rising order with the evens separated from the odds. Now to aid our calculation we turn to Riordan.

Theorem 5 (Riordan) The number of combinations of 0 through $n-1$ such that each combination in rising order has i successions of rise 1 is

$$f_i(n,k) = \binom{k-1}{i}\binom{n-k+1}{k-i}.$$

Let $g_i(p,q,k)$ be the number of combinations of k numbers taken from the first p evens and the first q odds such that in total the number of successions of rise two in evens taken in order plus the number of successions of rise two in the odds taken in rising order is i. Clearly when $p = \lceil \frac{n}{2} \rceil$ and $q = \lfloor \frac{n}{2} \rfloor$, g is the quantity we seek above. Note we are picking a total of k things. Of these l are even numbers and $k-l$ are odd. In total there are i successions. Of these j come from the evens and $i-j$ from the odds. Clearly the number of combinations of p evens taken l at a time that have j successions is $f_j(p,l)$. Multiplying this by the $f_{i-j}(q,k-l)$ ways to pick the odds will give us our total number of ways to pick our elements. Now all we have to do is sum over all the ways of distributing our picks and our successions between the evens and odds. That is,

$$g_i(p,q,k) = \sum_{j=0}^{i}\sum_{l=0}^{k} f_j(p,l) f_{i-j}(q,k-l).$$

n/x	0	1	2	3	4	5	6	7
3	6							
4	24							
5	114	6						
6	674	44	2					
7	4714	294	30	2				
8	37754	2272	276	16	2			
9	340404	20006	2236	216	16	2		
10	3412176	193896	20354	2200	156	16	2	
11	37631268	2056012	206696	20738	1908	160	16	2

Table 1: $s_1^2(n, x)$

This formula breaks down in the case $i = k = 0$ in which case $g_0(p, q, 0) = 0$. Now we let

$$b(n, k) = \sum_{i=0}^{k-1} g_i \left(\left\lceil \frac{n-4}{2} \right\rceil, \left\lfloor \frac{n-4}{2} \right\rfloor, k \right) (n - 2k + i)!$$

Then we can conveniently summarize our result by

Theorem 6 For $n \geq 3$ and $x \geq 0$,

$$s_1^2(n, x) = \delta_{x,0} n! + \sum_{k=1}^{n-4} (-1)^{k+x} \binom{k}{x} b(n, k).$$

Now to calculate $c_1^1(n, x)$ we can mirror Riordan almost perfectly except the number of permutations that have a certain selection of k progressions is $(n - 2k + i - 1)!$. Where as usual i is the number of successions of rise 1 in the starting elements of the progressions taken in order. Letting

$$a^{\circ}(n, k) = \sum_{i=0}^{k-1} f_i(n - 2, k)(n - 2k + i - 1)!$$

we have

Theorem 7 For $n \geq 3$ and $x \geq 0$,

$$c_1^1(n, x) = \delta_{x,0}(n - 1)! + \sum_{k=1}^{n-2} (-1)^{k+x} \binom{k}{x} a^{\circ}(n, k).$$

n/x	0	1	2	3	4	5	6	7	8	9
3	1	1								
4	5	0	1							
5	20	3	0	1						
6	102	14	3	0	1					
7	627	72	17	3	0	1				
8	4461	468	87	20	3	0	1			
9	36155	3453	582	103	23	3	0	1		
10	328849	28782	4395	704	120	26	3	0	1	
11	3317272	267831	37257	5435	834	138	29	3	0	1

Table 2: $c_1^1(n,x)$

We now turn our attention to $c_1^2(n,x)$. This case mirrors the calculation of $s_1^2(n,x)$ with similar modifications as were made to calculate $c_1^1(n,x)$. That is, we replace $(n-2k+i)!$ with $(n-2k+i-1)!$. Then letting

$$b^\circ(n,k) = \sum_{i=0}^{k-1} g_i\left(\left\lceil\frac{n-4}{2}\right\rceil, \left\lfloor\frac{n-4}{2}\right\rfloor, k\right)(n-2k+i-1)!$$

we get

Theorem 8 For $n \geq 3$ and $x \geq 0$,

$$c_1^2(n,x) = \delta_{x,0}(n-1)! + \sum_{k=1}^{n-4}(-1)^{k+x}\binom{k}{x} b^\circ(n,x).$$

The case for $\overline{s}_1^1(n,x)$ becomes somewhat more complicated. Here we have a total of n different progressions. However they cannot all appear simultaneously in a single permutation. The identity permutation has $n-2$ progressions. Note that this is the maximum. However, using our previous argument, if we looked for a permutation with the progressions 012, 123,...,$(n-2)(n-1)0$, then we would conclude that there are $(n-2(n-1)+(n-2))! = 0! = 1$ of them where now i counts modular successions. This however is wrong. There are no permutations that fit this qualification.

Lemma 9 There is no permutation that contains k modular progressions of rise and distance 1 with i modular successions of rise 1 in the initial elements of the k progressions when $n-2k+i \leq 0$.

n/x	0	1	2	3	4	5	6	7	8
3	2								
4	6								
5	22	2							
6	109	10	1						
7	657	55	7	1					
8	4625	356	54	4	1				
9	37186	2723	362	44	4	1			
10	336336	23300	2837	368	34	4	1		
11	3379058	220997	25408	2967	330	35	4	1	
12	37328103	2308564	249736	26964	3100	292	36	4	1

Table 3: $c_1^2(n, x)$

Proof Let us consider k modular progressions with i modular successions of rise 1 in the initial elements of the k progressions. The *span,* s, of the progressions is the length of the shortest sequence that can contain all k progressions. A *block* of progressions are those progressions whose initial terms form successions of rise 1. Note that the span of a block is two more than the number of progressions in the block.

Let b be the number of blocks that the k progressions form. Note that $b = k - i$ and two blocks can overlap in at most one element. Ordering the blocks by the smallest initial value of the progression in the block, let k_j be the number of progressions in the j^{th} block and so the span of the j^{th} block is $k_j + 2$. The span s of the k progressions is not $\sum(k_j + 2)$ since blocks can overlap in one position. With b blocks there are at most $b - 1$ overlaps and so

$$s \geq \sum_{j=1}^{b}(k_j + 2) - (b - 1) = k + 2b - b + 1 = k + (k - i) + 1 = 2k - i + 1.$$

If $n - 2k + i \leq 0$, then $n \leq 2k - i < s$, a contradiction. ■

Thus we define a weight function which will replace the factorial in our formula.

$$w(m) = \begin{cases} 0 & : m \leq 0 \\ m! & : \text{otherwise} \end{cases}$$

We now seek the number of combinations of the numbers 0 through $n-1$ taken k at a time that have i modular successions when taken in order and circularized (if we did not circularize we could not account for the good matching of $(n-2)(n-1)0$ and 012). We will call this number $\overline{g}_i(n,k)$. Note that when we choose our k integers they will form successive blocks with a block of length l having $l-1$ successions. Thus we know that if we have i successions then we must have $k-i$ blocks. Thus the question becomes how many ways are there to split the numbers 0 through $n-1$ into $k-i$ blocks when we write them circularly. Note that the number of compositions of n items into k parts where each part has to have at least one item is $\binom{n-1}{k-1}$. We will denote this $p(n,k)$. Note that we have a composition of the k chosen numbers into $k-i$ parts and the $n-k$ unchosen numbers into $k-i$ parts. However we cannot simply take the product of $p(k,k-i)$ and $p(n-k,k-i)$ since this would assume that 0 is at the beginning of a block of chosen elements. However this is one case. Another case is that 0 is at the beginning of a block of unchosen elements. In which case we again have $p(k,k-i)p(n-k,k-i)$. If 0 is chosen but not at the beginning of a block, then we form a composition consisting of $k-i+1$ parts where the part at the end meets up with the part at the beginning. Then there are $p(k,k-i+1)p(n-k,k-i)$ ways to do this. If 0 is not chosen and not at the beginning of a block we split the unchosen elements into $k-i+1$ parts giving us $p(k,k-i)p(n-k,k-i+1)$ combinations that fit this description. Then we have

$$\begin{aligned}\overline{f}_i(n,k) &= 2p(k,k-i)p(n-k,k-i)+p(k,k-i+1)p(n-k,k-i)+\\ &\quad p(k,k-i)p(n-k,k-i+1)\end{aligned}$$

with boundary conditions that $\overline{f}_0(n,0)=\overline{f}_n(n,n)=1$. Now, as before, we define

$$\overline{a}(n,k)=\sum_{i=0}^{k-1}\overline{f}_i(n,k)w(n-2k+i)$$

giving us

Theorem 10 For $n \geq 3$ and $x \geq 0$,

$$\overline{s}_1^1(n,x)=\delta_{x,0}n!+\sum_{k=1}^{n}(-1)^{k+x}\binom{k}{x}\overline{a}(n,k).$$

We do not include a table for these values since by our general theorems it suffices to multiply the elements in the table for $c_1^1(n,x)$ by their row number.

Turning our attention to $\overline{s}_1^2(n,x)$, the situation becomes again more complicated. Note though, if n is odd we can multiply by the inverse of 2 modulo n to reduce to the $r=d=1$ case. Because of this we only need to consider the case when n is even. Determining which combinations of progressions are unallowed becomes more difficult. We cannot have "wrap around" progressions in either evens or odds. That is there are no permutations containing 024, 246, ..., $n02$ or any other combination of progressions that cover either the evens or the odds. We then cannot have either too many progressions in either the evens or odds. We define a more complicated weight function than before.

$$w_2(p,q,j,j',l,l') = \begin{cases} 0 & : p-2l+j \le 0 \text{ or } q-2l'+j' \le 0 \\ (p+q-2(l+l')+j+j')! & : \text{otherwise} \end{cases}$$

Where p is the total number of even progressions taken l at a time and forming j successions of rise 2 in their initial elements. The variables q, l', and j' refer to the odd progressions. Clearly we are going to have to apply this weight function at a different point in our function where we still have the information about number of even and odd progressions. Like $s_1^2(n,x)$ we will sum over all the different ways to distribute our picks and progressions between the evens and odds. We define

$$\overline{g}_i(p,q,k) = \begin{cases} w(p+q-2k+i)\overline{f}_i(p+q,k) & : p+q \text{ is odd} \\ \sum_{j=0}^{i}\sum_{l=0}^{k} w_2(p,q,j,i-j,l,k-l)\overline{f}_j(p,l)\overline{f}_{i-j}(q,k-l) & : \text{otherwise} \end{cases}$$

This then is the number of permutations consisting of p even elements and q odd that have at least k progressions forming i successions of rise 2 in their initial elements. We then let

$$\overline{b}(n,k) = \sum_{i=0}^{k-1} \overline{g}_i\left(\left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor, k\right).$$

Theorem 11 For $n \ge 3$ and $x \ge 0$,

$$\overline{s}_1^2(n,x) = \delta_{x,0}n! + \sum_{k=1}^{n} (-1)^{k+x}\binom{k}{x}\overline{b}(n,k).$$

n/x	0	1	2	3	4	5	6	7
3	3	3						
4	24	0						
5	100	15	0	5				
6	594	108	18	0				
7	4389	504	119	21	0	7		
8	35744	3520	960	64	32	0		
9	325395	31077	5238	927	207	27	0	9
10	3288600	288300	42050	8800	900	100	50	0

Table 4: $\overline{s}_1^2(n,x)$

Next on our list is $\overline{c}_1^1(n,x)$. Here no combination of progressions is disallowed. However when $n = k = i$ our regular weight formula would say that there are $(n-2n+n-1)! = (-1)! = 0$ permutations of this type. This however is incorrect. There is one permutation that has all n progressions. These n progressions form n successions. This permutation is the singular circular identity. Then our weight function is

$$w^\circ(i,n,k) = \begin{cases} 1 & : i = k = n \\ (n-2k+i-1)! & : \text{otherwise} \end{cases}$$

Also note that our f's will be precisely the same as for $\overline{s}_1^1(n,x)$. If

$$\overline{a}^\circ(n,k) = \sum_{i=0}^{k} w(i,n,k)\overline{f}_i(n,k),$$

then we have

Theorem 12 For $n \geq 3$ and $x \geq 0$,

$$\overline{c}_1^1(n,x) = \delta_{x,0}(n-1)! + \sum_{k=0}^{n} (-1)^{k+x}\binom{k}{x}\overline{a}^\circ(n,k).$$

Finally we have $\overline{c}_1^2(n,x)$. Note first that if n is odd we can again reduce to $\overline{c}_1^1(n,x)$. We will then concentrate on n even. This precisely mirrors the case of $\overline{s}_1^2$ except that our weight function now refers to circular permutations so

n/x	0	1	2	3	4	5	6	7	8	9	10
3	1	0	0	1							
4	5	0	0	0	1						
5	18	5	0	0	0	1					
6	95	18	6	0	0	0	1				
7	600	84	28	7	0	0	0	1			
8	4307	568	116	40	8	0	0	0	1		
9	35168	4122	810	156	54	9	0	0	0	1	
10	321609	33910	5975	1100	205	70	10	0	0	0	1

Table 5: $\overline{c}_1^1(n,x)$

we will need to reduce the factorial by 1. Then

$$w_2^\circ(p,q,j,j',l,'l) = \begin{cases} 0 & : p-2l+j \leq 0 \text{ or } q-2l'+j' \leq 0 \\ (p+q-2(l+l')+j+j'-1)! & : \text{otherwise} \end{cases}$$

where the arguments are the same as defined in w_2. Then we also have

$$\overline{g}_i^\circ(p,q,k) = \begin{cases} w^\circ(i,n,k)\overline{f}_i(n,k) & : p+q \text{ is odd} \\ \sum_{j=0}^{i}\sum_{l=0}^{k} w_2^\circ(p,q,j,i-j,l,k-l)\overline{f}_j(p,l)\overline{f}_{i-j}(q,k-l) & : \text{ otherwise} \end{cases}$$

and

$$\overline{b}^\circ(n,k) = \sum_{i=1}^{k-1} \overline{g}_i^\circ\left(\left\lceil\frac{n}{2}\right\rceil, \left\lfloor\frac{n}{2}\right\rfloor, k\right).$$

Using these we have

Theorem 13 For $n \geq 3$ and $x \geq 0$,

$$\overline{c}_1^2(n,x) = \delta_{x,0}(n-1)! + \sum_{k=1}^{n}(-1)^{k+x}\binom{k}{x}\overline{b}^\circ(n,k).$$

Note that by our general theorems we also get $r=1$ and $d=2$ for each of our functions.

6 Conclusion

Clearly there is almost an unlimited number of questions you could still ask on this subject. One of the most interesting is whether or not there is a

n/x	0	1	2	3	4	5	6	7	8
3	1	0	0	1					
4	6	0	0	0					
5	18	5	0	0	0	1			
6	93	18	9	0	0	0			
7	600	84	28	7	0	0	0	1	
8	4320	512	192	0	16	0	0	0	
10	321630	34000	5625	1400	200	0	25	0	
12	36199458	3178656	457524	66168	13014	1656	288	0	36

Table 6: $\overline{c}_1^2(n,x)$

polynomial time algorithm for calculating how many permutations have no progressions of any rise or distance. It is unlikely that the techniques used in this thesis will be able to be pushed much further. Many of the arguments relied on considering odds and evens separately. If you wanted to analyze $r = 3$ you would likely need three cases. Generating functions seem like they might be a profitable line of attack. Also it is possible that these techniques could be pushed to calculate the $r = d = 2$ case.

7 References

[1] J. Riordan, Permutations without 3-Sequences, *Bull. Amer. Math. Soc.* **51** (1945), 745–748.

[2] W. M. Dymàček and I. Lambert, Circular permutations avoiding runs of i, $i+1$, $i+2$ or i, $i-1$, $i-2$, *J. Integer Seq.* **14** (2011), Article 11.1.6.