Realizability of $n$-Vertex Graphs with Prescribed Vertex Connectivity, Edge Connectivity, Minimum Degree, and Maximum Degree

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It is true, in graph theory, that for any graph, $G$, $\kappa \leq \lambda \leq \delta \leq \Delta$. However, this only shows that the non-decreasing sequence exists for every graph. It does not describe what the values of $\kappa$, $\lambda$, $\delta$, $\Delta$ are for a given graph, $G$, or whether or not the graph is realizable when values for each variable are chosen at random. This paper aims to find the smallest number of vertices on which a graph with these specified parameters is realizable. In other words, for which values of $\kappa$, $\lambda$, $\delta$, and $\Delta$ is the graph realizable. Once the smallest number of vertices for a realizable graph is determined, it is then possible to generalize cases to find how large the graph can be. From there, a general picture of the graph, $G$, can be used to verify and prove that all parameters have been realized.

**Introduction**

Since 1972, there has been extensive research on the realizability of graphs. The parameters considered have varied considerably. In 1979, F.T. Boesch and C.L. Suffel were able to generalize realizability for graphs on $n$ vertices, with vertex connectivity $\kappa$, minimum degree $\delta$ and maximum degree $\Delta$ [1]. They found that the quadruple $(n, \kappa, \delta, \Delta)$ is realizable if and only if one of the following five mutually exclusive conditions holds:

(I) $\delta < \lfloor \frac{1}{2} n \rfloor$ :

(i) $0 = \kappa \leq \Delta \leq n - \delta - 2$ and if $\delta = \Delta$, then $n\delta$ is even;

(ii) $1 \leq \kappa \leq \delta < \Delta \leq n - 1$;

(iii) $1 \leq \kappa \leq \delta = \Delta < \lfloor \frac{1}{2} n \rfloor$, $n\delta$ is even, and if $\kappa = 1$, then $2 < \delta < \frac{1}{2} n - 1$;

(II) $\delta \geq \lfloor \frac{1}{2} n \rfloor$ :

(i) $1 \leq 2\delta - n + 2 \leq \kappa \leq \delta < n - 1$ and $\kappa \Delta \geq (n - \kappa)\kappa - (\kappa - (2\delta - n + 2))(n - \delta - 1)$;

(ii) $\kappa = \delta = \Delta = n - 1$.

Furthermore, if any three of the parameters $n$, $\kappa$, $\delta$, and $\Delta$, are given, it is possible to calculate the complete range of possible values for the unspecified parameter. As a corollary, Boesch and Suffel were able to find the minimum value of $\kappa$ among all $(n, \delta, \Delta)$ graphs.

1. For $\delta < \lfloor \frac{1}{2} n \rfloor$ and $\Delta \leq n - \delta - 2$,
   \[
   \min \kappa(n, \delta, \Delta) = 0
   \]

2. For $\delta < \lfloor \frac{1}{2} n \rfloor$ and $\Delta > n - \delta - 2$,
   \[
   \min \kappa(n, \delta, \Delta) = 1
   \]

3. For $n - 1 \neq \delta \geq \lfloor \frac{1}{2} n \rfloor$,
   \[
   \min \kappa(n, \delta, \Delta) = \min\left(\frac{1}{2}(-(\Delta - \delta - 1) + \sqrt{(\Delta - \delta - 1)^2 + 4(n - \delta - 1)(2\delta - n + 2)})\right)
   \]

4. For $\delta = n - 1$
   \[
   \min \kappa(n, \delta, \Delta) = n - 1
   \]

A year later, Boesch and Suffel were able to generalize realizability for graphs with a prescribed number of vertices, $n$, edge connectivity, $\lambda$, minimum degree, $\delta$, and maximum degree, $\Delta$ [2]. A quadruple of non-negative integers $(n, \lambda, \delta, \Delta)$ is realizable if and only if exactly one of the following conditions hold:
As a corollary, they found that a quadruple of non-negative integers \((n, \lambda, \delta, \Delta)\) is realizable if and only if either

\((I)\lambda = \delta\)

and (A) If \(\delta = 0\), then

\((1)0 < \Delta \leq n - 2\) or,
\((2)0 = \Delta \leq n - 2\)

and (B) If \(\delta = 1\), then

\((1)n = 2\) and \(\lambda = \Delta = 1\) or,
\((2)n > 2\) and \(\delta < \Delta \leq n - 1\)

and (C) If \(\delta > 1\), then

\((1)\delta = \Delta\) and \(n\Delta\) is even and \(\Delta \leq n - 1\) or,
\((2)\delta < \Delta\) and \(\Delta \leq n - 1\);

or \((II)\lambda < \delta\)

and (A) If \(\delta = \Delta\) then \(\Delta \leq \lfloor \frac{n}{2} \rfloor - 1\), \(n\Delta\) is even, and \(\Delta\) even or \(\Delta = \frac{n}{2} - 1 \Rightarrow \lambda\) is even

and (B) If \(\delta < \Delta\) then \(\Delta \leq n - 3\), and \(\delta \leq \min((\lfloor \frac{n}{2} \rfloor - 1), (n - 2 - \Delta + \lambda))\).

Similarly, if any three of the parameters \(n, \lambda, \delta, \Delta\), are given, it is possible to calculate the complete range of possible values of the unspecified parameter. As with their previous research, Boesch and Suffel were able to develop a corollary defining the minimum value of \(\lambda\) among all \((n, \delta, \Delta)\) graphs.

Our research focuses on the realizability of the quadruple of non-negative integers \((\kappa, \lambda, \delta, \Delta)\) where \(\kappa\) is vertex connectivity, \(\lambda\) is edge connectivity, \(\delta\) is minimum degree, and \(\Delta\) is maximum degree. Once realizability is determined, we aim to find the smallest set of vertices \(n\) for which the graph is realizable, and from there, generalize to find the set of all possible \(n\) values.

Definitions

We will begin by formally defining a graph and the related concepts necessary for discussing the realizability of graphs. Here, we also describe the notation that will be used throughout this paper.

A **graph**, \(G\), consists of two sets, \(V\) and \(E\), such that \(V\) is a finite, non-empty set and the elements of \(E\) are two-element subsets of \(V\). The elements of \(V\) are called vertices and the elements of \(E\) are called edges.

The vertices, \(u\) and \(v\), are **adjacent** if \([u, v] \in E\). The edge connecting \(u\) to \(v\) is written as \(e_{u,v}\). Then the vertices are **incident** to the edge \(e_{u,v}\). Two edges are called adjacent if they are incident to the same vertex. The set of edges connecting a vertex \(u\) to a set of vertices \(S\) is denoted \(e_{u,S}\), and the set of edges connecting two disjointed sets of vertices \(S\) and \(T\) is denoted \(e_{S,T}\).
A sequence of not necessarily distinct edges \((e_1, e_2, ..., e_n)\) is called a \textbf{walk}, if \(e_i\) and \(e_{i+1}\) are adjacent for \(1 \leq i < n\). The walk is directed from one vertex to another \((v_0, v_1, ..., v_n)\), where \(e_i = e_{v_{i-1}v_i}\) for \(1 \leq i < n\). The \textit{initial vertex} is \(v_0\) and the \textit{final vertex} is \(v_n\). If \(v_0 = v_n\), then the walk is \textit{closed}. A closed walk with distinct edges and vertices is called a \textbf{circuit}.

A graph, \(G\), is \textbf{connected} if, for every pair of vertices \(u, v \in G\), there exists a walk from \(u\) to \(v\). A graph that is not connected is called \textbf{disconnected}. A set of vertices in a disconnected graph which are connected by a walk to each other but not to any other vertex in the graph is called a \textbf{component} of \(G\).

A \textbf{separating set} of \(G\) is a set of vertices whose removal disconnects \(G\). If the separating set has only one vertex, then the vertex is a \textbf{cut vertex}. If \(G\) is connected, then the \textit{connectivity} of \(G\), denoted \(\kappa(G)\) or simply \(\kappa\), is the size of the smallest separating set of \(G\). If \(G\) is not connected, \(\kappa(G) = 0\).

Similarly \textbf{disconnecting set} of \(G\) is a set of edges whose removal disconnects \(G\). If the disconnecting set contains only one edge, then that edge is called a \textbf{bridge}. If \(G\) is connected, the the \textit{edge-connectivity} of \(G\), denoted \(\lambda(G)\) or simply \(\lambda\), is the size of the smallest disconnecting set of \(G\). If \(G\) is disconnected, \(\lambda(G) = 0\). Let \(S\) be the set of \(\lambda\) edges such that \(G - S\) is disconnected.

The \textit{degree} of a vertex \(v\) denoted \(\deg(v)\), is the number of edges incident to \(v\). The \textbf{minimum degree} of a graph, denoted by \(\delta(G)\) or simply \(\delta\), is the smallest degree of any vertex in the graph. The \textbf{maximum degree} of a graph, denoted by \(\Delta(G)\) or simply \(\Delta\), is the largest degree of any vertex in the graph. If \(\delta = \Delta\), then the graph is \textit{regular} of degree \(\delta\) [3].

\textbf{Note:} We will define \(\kappa(K_n) = n - 1\), where \(K_n\) is the complete graph on \(n\) vertices. Therefore, \(\kappa(K_1) = \lambda(K_1) = \delta(K_1) = \Delta(K_1) = 0\).

A non-decreasing sequence of numbers \((\kappa, \lambda, \delta, \Delta)\) is \textbf{realizable} if there exists a graph with connectivity \(\kappa\), edge-connectivity \(\lambda\), minimum degree \(\delta\), and maximum degree \(\Delta\).

Let \(G\) be a graph of \(n\) vertices. If, for every pair of vertices \(u, v \in G\), \(e_{u,v} \in E\), then the graph is a \textbf{complete graph} and is denoted by \(K_n\).

\textbf{Notation}

1. \(F_{\kappa,\lambda,\delta,\Delta}\) is the set of integers, \(n\), for which there is a graph, \(G\), with \(n\) vertices for which \(\kappa(G) = \kappa, \lambda(G) = \lambda, \delta(G) = \delta,\) and \(\Delta(G) = \Delta\).

2. Let \(n_{\kappa,\lambda,\delta,\Delta}\) denote the smallest number of vertices for which \(G\) is realizable.

3. If \(L\) is a set of vertices of \(G\), we denote \(|L|\) by \(n_L\).

4. Let \(S\) be the set of edges with \(\lambda\) edges such that \(G - S\) is not connected. We let \(G - S = L \cup M\) with \(|L| \leq |M|\).

5. Let \(L\) and \(M\) be graphs with \(U\) a set of vertices of \(L\) and \(V\) a set of vertices of \(M\). The graph \(G = L \cup M\) has vertex set \(V(L) \cup V(M)\) and edge set \(E(L) \cup E(M) \cup \{e_{i,j} : i \in U, j \in V\}\).

6. Let \(n\) and \(k\) be positive integers. The Harary graph, \(H_{n,k}\) has vertices \(\{0, 1, ..., n - 1\}\). If \(k\) is even, then the edge set of \(H_{n,k}\) is \(\{\{v, v + i\} : 0 < i \leq \frac{n}{2}\}\). If \(k\) is odd and \(n\) is even, then \(H_{n,k} = H_{n,k-1}\) with the additional edges \(\{\{v, v + \frac{n}{2}\} : 0 \leq v < \frac{n}{2}\}\). If both \(k\) and \(n\) are odd, then \(H_{n,k} = H_{n,k-1}\) with the additional edges \(\{\{v, v + \frac{n+1}{2}\} : 0 \leq v < \frac{n-1}{2}\}\) and the edge \(\{0, \frac{n-1}{2}\}\).

7. Let \(n\) and \(k\) be positive integers. The graph, \(H_{n,k}^-\) is a Harary graph without the edge \(\{0, \frac{n-1}{2}\}\).
(8) For a graph $G = L \cup M$ with $u \in G$, let $\deg_L(u)$ denote the degree of the vertex $u$ in component $L$. In other words, $\deg_L(u)$ is the number of vertices in $L$ that are adjacent to $u$. Now, we will introduce some preliminary theorems and observations necessary for realizability.

**Theorems: Generic**

(1) **Theorem 1:** (Whitney [1932]) For any graph, $G$, $\kappa(G) \leq \lambda(G) \leq \delta(G) \leq \Delta(G)$.

(2) **Observation 1:** If a graph has a vertex of degree $k$, then the graph has at least $k + 1$ vertices.

(3) **Theorem 2:** If a graph is connected but not complete, then $n \geq 2\delta - \kappa + 2$.

**Proof.** Let $G - S = L \cup M$ where $S$ is a set of vertices of size $\kappa$. If $u \in L$, then $\delta \leq \deg_L(u) + \deg_S(u) \leq \deg_L(u) + \kappa$. Hence $\delta - \kappa \leq \deg_L(u)$ and so $n_L \geq \delta - \kappa + 1$. Since this is also true for $M$, $n_G = n_L + n_M + n_S \geq (\delta - \kappa + 1) + (\delta - \kappa + 1) + \kappa = 2\delta - \kappa + 2$. \hfill \Box

(4) **Observation 2:** For all graphs, $n \geq \Delta + 1$.

(5) **Theorem 3:** If $\delta \geq \lceil \frac{n}{2} \rceil$, then $\lambda = \delta$. [4]

**Theorems: $\lambda < \delta$**

(1) **Theorem 1:** If $\lambda < \delta$, then $n \geq 2\delta + 2$.

Note that this is larger than the generic bound from Theorem 2 above.

**Proof.** Let $G - S = L \cup M$ where $S$ is a set of $\lambda$ edges. Let $u \in L$. So $\deg_L(u) + \deg_S(u) = \deg(u) \geq \delta$. Hence $\deg_S(u) \geq \delta - \deg_L(u)$. Thus

$$\delta > \lambda$$

$$= \sum_{u \in L} \deg_S(u)$$

$$\geq \sum_{u \in L} (\delta - \deg_L(u))$$

$$= n_L \cdot \delta - \sum_{u \in L} \deg_L(u).$$

Thus

$$\sum_{u \in L} \deg_L(u) > n_L \cdot \delta - \delta$$

$$= \delta(n_L - 1)$$

Also, $\frac{1}{2} \sum_{u \in L} \deg_L(u) = m_L \leq \binom{n_L}{2}$ or $\sum_{u \in L} \deg_L(u) \leq n_L(n_L - 1)$. Therefore $n_L(n_L - 1) > \delta(n_L - 1)$ or $n_L > \delta$. Hence $n = n_L + n_M \geq n_L + n_L \geq \delta + 1 + \delta + 1 = 2\delta + 2$. \hfill \Box
(2) **Corollary 1:** If \( \lambda < \delta \), then removing any edge cut leaves two components, each with at least \( \delta + 1 \) vertices.

(3) **Theorem 2:** If \( \lambda < \delta \), then \( n \geq \Delta + \delta + \kappa - \lambda + 1 \).

**Proof.** Let \( S \) be an edge cut of \( G \) and \( G - S = L \cup M \). From Corollary 1 in this section, \( n_L \geq \delta + 1 \) and \( n_M \geq \delta + 1 \). Since \( n = n_L + n_M \), both \( n_L \) and \( n_M \) are less than or equal to \( n - (\delta + 1) \).

Let \( v \in G \). This argument will work for either component of \( G - S, L \) or \( M \), and so we will use \( M \). Let \( T = \{ v \in M : v \text{ is an endpoint of an edge in } S \} \). Since \( |T| \leq |S| = \lambda \) and \( \lambda < \delta \), then \( T \not\subseteq M \). Also note that \( G - T \) has vertices in both \( L \) and \( M \) since removing \( T \) from \( G \) also removes \( S \) from \( G \). Since for \( G - T \) is not connected. Thus \( T \) is a separating set of \( G \) and so \( |T| \geq \kappa \).

If \( v \) is not in \( T \), then \( \deg_L(v) = 0 \) and so \( \lambda - \kappa + 1 > \deg_L(v) \). If \( v \in T \), then since

\[
\lambda = \sum_{x \in T} \deg_S(x) \\
\geq \deg_S(v) + |T| - 1 \\
\geq \deg_L(v) + \kappa - 1
\]

\( \lambda - \kappa + 1 \geq \deg_L(v) \). So

\[
\deg(v) = \deg_L(v) + \deg_M(v) \\
\leq \lambda - \kappa + 1 + n_M - 1 \\
\leq \lambda - \kappa + n - (\delta + 1).
\]

Since there is a \( v \in G \) with \( \deg(V) = \Delta, \Delta \leq \lambda - \kappa + n - \delta - 1 \) or \( n \geq \Delta + \delta + \kappa - \lambda + 1 \), as desired. \( \square \)

**Theorems: Regular Graphs**

(1) **Theorem 1:** If \( G \) is regular of even degree, then \( \lambda \) is even. Contrapositive: If \( G \) is regular and \( \lambda \) is odd, then \( \delta \) is odd.

**Proof.** Let \( G \) be regular, let \( S \) be an edge cut, and let \( G - S = L \cup M \). Let \( u \in L \). Thus \( \deg(u) = \deg_L(u) + \deg_S(u) \) or \( \deg_L(u) = \deg(u) - \deg_S(u) \). Hence

\[
\sum_{u \in L} \deg_L(u) = \sum_{u \in L} (\deg(u) - \deg_S(u)) \\
= \sum_{u \in L} \deg(u) - \sum_{u \in L} \deg_S(u) \\
= n_L \cdot \delta - |S| \\
= n_L \cdot \delta - \lambda
\]
Since $\sum_{u \in L} \deg_L(u)$ is even, $\lambda$ and $n_L \cdot \delta$ have the same parity. Thus if $\delta$ is even, then $\lambda$ is even. \qed

(2) Corollary 1: If $G$ is regular and $n_L$ is the size of a component of $G$ after removing an edge cut, then $n_L$, $\delta$, and $\lambda$ have the same parity.

(3) Theorem 2: If $G$ is regular and if $\delta$ and $\lambda$ are both odd with $\lambda < \delta$, then $n$ is even and $n \geq 2\delta + 4$.

Proof. From the previous proof, we know that $n_L \cdot \delta - \lambda$ is even. Thus under our hypotheses, $n_L$ must be odd. By the Corollary in the $\lambda < \delta$ section, $n_L \geq \delta + 1$. Since $\delta$ is odd, $n_L \geq \delta + 2$ and so $n \geq 2\delta + 4$. \qed

Note: For $\kappa = \lambda = \delta = \Delta$, $K_\kappa$ is the smallest graph realizing these parameters. In general, the Harary graph $H_{n,\delta}$ realizes $\kappa = \lambda = \delta = \Delta$, where $n \geq \kappa$.

Realizability of $\kappa = 0$

Now we can prove the realizability of a graph $G$, where $\kappa(G) = \kappa, \lambda(G) = \lambda, \delta(G) = \delta, \Delta(G) = \Delta$, for the case where $0 = \kappa = \lambda < \delta \leq \Delta$.

Theorem: If $\kappa = 0$ then,

$$F_{\kappa,\lambda,\delta,\Delta} = \{n \in \mathbb{N} : n \geq \delta + \Delta + 2\}$$

unless $\delta \Delta = 0$, in which case $F_{0,0,0,0} = \mathbb{N}$ or $\delta \Delta$ is odd and $\delta = \Delta$, in which case $F_{0,0,\delta,\Delta} = \{n \in \mathbb{N} : n \geq 2\delta + 2\}$.

Hence $n_{\kappa,\lambda,\delta,\Delta} = \delta + \Delta + 2$ unless $\delta \Delta = 0$ in which case $n_{0,0,0,0} = 1$.

Proof. Except for $K_1$, if $\kappa = 0$, then any graph realizing $(0,0,\delta,\Delta)$ must be disconnected and thus must have a vertex of degree $\Delta$ in a component. That component must have at least $\Delta + 1$ vertices. Any other component must have a vertex of degree at least $\delta$ and hence that component must have at least $\delta + 1$ vertices. Thus the graph must have at least $\delta + \Delta + 2$ vertices.

Note that $N_n$, the null graph on $n$ vertices, has all four parameters $0$ and hence $F_{0,0,0,0} = \mathbb{N}$. We now assume that $\Delta > 0$.

Let $G_1 = K_{\delta + 1} \cup H_{n-\delta-1,\Delta}$ and $G_2 = H_{n-\Delta-1,\delta} \cup K_{\Delta+1}$. Note that if $n = \delta + \Delta + 2$ then $G_1 = G_2 = K_{\delta + 1} \cup K_{\Delta+1}$.

If $\Delta$ or $n - \delta - 1$ are even, then $G_1$ has $\delta(G_1) = \delta, \Delta(G_1) = \Delta$, and $n(G_1) = n$. Note that $n - \delta - 1 > \Delta$ and so $n > \delta + \Delta + 1$.

If $\Delta$ and $n - \delta - 1$ are odd and $\delta$ is even, then $n - \Delta - 1$ is even. Thus, $\delta(G_2) = \delta, \Delta(G_2) = \Delta$, and $n(G_2) = n$. Note that $n = \Delta - 1 > \delta$ and so $n > \delta + \Delta + 1$.

Finally, suppose that $\Delta, \delta$, and $n - \delta - 1$ are all odd. Hence $n$ is also odd and so $G$ cannot be regular. Thus $\delta < \Delta$ and so $\delta(H_{n-\Delta-1,\delta} = \delta$ and $\Delta(H_{n-\Delta-1,\delta} = \delta + 1 \leq \Delta$. Thus $\delta(G_2) = \delta$ and $\Delta(G_2) = \Delta$.

If $\delta = \Delta$ are both odd, then $n$ must be even and we again use $G_1$. \qed
Realizability of $\kappa = 1$

Now consider the graph, $G$, where $1 = \kappa \leq \lambda \leq \delta \leq \Delta$. We will prove several necessary conditions in this case before proving the conditions for realizability.

(1) **Theorem 1:** For $n > 2$, if $\kappa = 1$, the $2\lambda \leq \Delta$.

**Proof.** Let $G$ be a graph with more than two vertices for which $\kappa(G) = 1$. Let $G - v = L \cup M$. Note that $e_{v,L}$ is a disconnecting set for $G$. Since $\deg_L(v) = |e_{v,L}|$, we have that $\deg_L(v) \geq \lambda$. Likewise, $\deg_M(v) \geq \lambda$. Thus

$$\Delta \geq \deg(v) = \deg_L(v) + \deg_M(v) \geq \lambda + \lambda = 2\lambda.$$  

(2) **Theorem 2:** If $1 = \kappa \leq \lambda < \delta < \Delta$ and $\delta + \lambda > \Delta$, then $n \geq 2\delta + 3$.

**Proof.** Let $w$ be a cut-vertex in $G$ and let $G - w = L \cup M$. Let $S = [w,L]$ and $T = [w,M]$. Note that $|S| \geq \lambda$ and $|T| \geq \lambda$. If $|S| \geq \delta$ or if $|T| \geq \delta$, then $\deg(w) \geq \delta + \lambda > \Delta$, a contradiction. Thus $\lambda \leq |S| < \delta$ and $\lambda \leq |T| < \delta$.

If for all $u \in L$, $\deg_L(u) = \delta - 1$, then $n_L \geq \delta$ and so $|S| = \delta$, a contradiction. Hence there is a $u \in L$ with $\deg(u) = \delta$. Thus $n_L \geq \delta + 1$. Since this same argument applies to $M$, $|M| \geq \delta + 1$. Therefore $n = n_L + n_M + 1 \geq 2\delta + 3$.  

(3) **Theorem 3:** If $1 = \kappa \leq \lambda < \delta = \Delta$, then $n \geq 2\delta + 3$ and if $\delta$ is odd, then $n \geq 2\delta + 4$.

**Proof.** Let $G$ be a regular graph with a cut-vertex, $w$, where $G - w = L \cup M$. If $\deg_L(w) = n_L$, then for any $u \in L$, $\deg_L(u) = \delta - 1$. So $\delta \leq n_L = \deg_L(w)$. This implies that $\deg_{G - L}(w) = 0$, a contradiction to $w$ being a cut-vertex. Thus there is a $u \in L$ not adjacent to $w$. Hence $\deg(u) = \deg_L(u) = \delta$ and thus $n_L \geq \delta + 1$. Thus is also true for $M$ and so $n \geq 2\delta + 3$.

If $\delta$ is odd, then $n$ must be even and therefore $n \geq 2\delta + 4$.  

Now that we have proven these conditions, we have the following statement concerning the realizability of a graph $G$ with $1 = \kappa \leq \lambda \leq \delta \leq \Delta$.

**Theorem:** If $\kappa = 1$, then

$$F_{1,\lambda,\lambda,\Delta} = \{ n \in \mathbb{N} : n \geq \Delta + 1 \}$$

unless $\Delta = 1$, in which case $F_{1,1,1,1} = 2$. For $\lambda < \delta < \Delta$ and $\lambda + \delta \leq \Delta$,

$$F_{1,\lambda,\lambda,\Delta} = \{ n \in \mathbb{N} : n \geq \Delta + \delta - \lambda + 2 \}.$$  

For $\lambda < \delta < \Delta$ and $\lambda + \delta > \Delta$ or for $\delta = \Delta$ and $\lambda$ all even,

$$F_{1,\lambda,\delta,\Delta} = \{ n \in \mathbb{N} : n \geq 2\delta + 3 \}$$  

For $\delta = \Delta$ odd,

$$F_{1,\lambda,\delta,\Delta} = \{ n \in \mathbb{E} : n \geq 2\delta + 4 \}.$$
Otherwise (for example, $2\lambda > \Delta$)

$$F_{1,\lambda,\delta,\Delta} = \emptyset.$$  

Hence $n_{1,\lambda,\lambda,\Delta} = \Delta + 1$, $n_{1,\lambda,\lambda,\Delta} = \Delta + \delta - \lambda + 2$ if $\lambda < \delta$ and $\lambda + \delta \leq \Delta$, $n_{1,\lambda,\delta,\Delta} = 2\delta + 3$ if $\lambda < \delta < \Delta$ and $\lambda + \delta > \Delta$ or $\lambda$ is even, and $\delta = \Delta$, and $n_{1,\lambda,\delta,\Delta} = 2\delta + 4$ if $\delta = \Delta$ is odd.

**Proof.** The only connected regular graph of degree 1 is $K_2$. Hence $F_{1,1,1,1} = \{2\}$.

For $\lambda = \delta$, note that for $n \geq 3$, $P_n$ has parameter set $(1,1,1,2)$ and so $F_{1,1,1,2}$ is as desired. In all other cases, let $G = K_{\lambda} \lor H_{n-\lambda,\Delta-\lambda}$ where the $\lambda$ edges from $K_{\lambda}$ are incident to any vertex, $u$, of degree $\Delta - \lambda$ in $H_{n-\lambda,\Delta-\lambda}$. Since $2\lambda \leq \Delta$, we have both that $\Delta - \delta \geq \delta$ and $\Delta - \lambda \geq 2$. So $H_{n-\lambda,\Delta-\lambda}$ has the correct connectivity and minimum degree. Also, $n - \lambda > \Delta - \lambda$ means that $n > \Delta$. Since each of the $\lambda$ vertices in $K_{\lambda}$ are connected to $u$ which has degree $\Delta - \lambda$ in $H_{n-\lambda,\Delta-\lambda}$, the degree of $u$ in $G$ is $\Delta$.

For $\lambda < \delta < \Delta$ and $\lambda + \delta \leq \Delta$, we know from Theorem 3 in the $\lambda < \delta$ section, that $n \geq \Delta + \delta + \kappa - \lambda + 1$ and since $\kappa = 1, n \geq \Delta + \delta - \lambda + 2$. Consider the graph $G = K_{\delta+1} \lor U H_{n-\delta-1,\Delta-\lambda}$ where we choose a vertex $v$ of degree $\Delta - \lambda$ in $H_{n-\delta-1,\Delta-\lambda}$ and $U$ is a set of any of the $\lambda$ vertices in $K_{\delta+1}$. Note that $H_{n-\delta-1,\Delta-\lambda}$ exists as long as $n - \delta - 1 > \Delta - \lambda$ or $n > \Delta + \delta - \lambda + 1$. Since $\delta < \lambda$, there is at least one vertex in $K_{\delta+1}$ of degree $\delta$ and since $\delta < \Delta$, no vertex in $K_{\delta+1}$ has degree larger than $\delta + 1 \leq \Delta$ in $G$. Note that again, $\deg(v) = \Delta$. Also, $\Delta - \lambda \geq \delta > \lambda$ and so $H_{n-\delta-1,\Delta-\lambda}$ has a minimum degree at least $\delta$ and edge connectivity at least $\lambda$.

For $\lambda < \delta < \Delta$ and $\lambda + \delta > \Delta$, we know from Theorem 2 in this $\kappa = 1$ section, that $n \geq 2\delta + 3$. Consider the graph $G_1 = K_{\delta+1} \lor U K_1 \lor V H_{n-\delta-2,\delta}$, where $U$ is a set of any $\lambda$ vertices in $K_{\delta+1}$ which we connect to $K_1 = \{u\}$ and $V$ is a set of any $\Delta - \lambda$ vertices in $H_{n-\delta-2,\delta}$ to connect to the $K_1$. Note that since $\delta > \Delta - \lambda$ and $n - \delta - 2 > \delta$, there are $\Delta - \lambda$ vertices of degree $\delta$ in $H_{n-\delta-2,\delta}$. This also means that $H_{n-\delta-2,\delta}$ exists for $n > 2\delta + 2$. Since $2\lambda \leq \Delta$, $\Delta - \lambda \geq \lambda$ and so the edge connectivity of $G_1$ is $\lambda$. Since $\lambda < \delta$, there is at least one vertex in $K_{\delta+1}$ of degree $\delta$ and so the minimum degree of $G_1$. Finally, $\deg(u) = \Delta$.

We now consider the regular case. Suppose $\delta$ is even and $\delta = \Delta$. By Theorem 1 in the Regular section, $\lambda$ also must be even. Consider $G_1$ from the previous case where $K_1 = \{u\}, U = \{0, 1, \ldots, \lambda-1\}$ are vertices in $K_{\delta+1}$, and $V = \{0, 1, \ldots, \Delta - \lambda - 1\}$ are vertices in $H_{n-\delta-2,\delta}$. Since $n > 2\delta + 2$, we have that $n - \delta - 2 > \Delta - \lambda - 1$ and so it is possible to choose $V$.

Since each vertex in $U$ and $V$ are adjacent to $u$, $\deg(u) = \delta$. Note, however, that the vertices in $U$ and $V$ have degree $\delta + 1$ and all other vertices have degree $\delta$. To make $G_1$ regular, delete the edges $e_{2i,2i+1}$ from both $K_{\delta+1}$ and $H_{n-\delta-2,\delta}$ where for $K_{\delta+1}, 0 \leq i < \frac{\lambda}{2}$ and for $H_{n-\delta-2,\delta}, 0 \leq i < \frac{\Delta-\lambda}{2}$. Since $2\lambda \leq \Delta = \delta$, removing fewer than $\lambda$ edges in either $K_{\delta+1}$ or $H_{n-\delta-2,\delta}$ leaves a connected graph. Since $\Delta - \lambda \geq \lambda$, removing fewer edges between $H_{n-\delta-2,\delta}$ and the $K_1$ or $K_{\delta+1}$ again leaves a connected graph. So this graph has edge connectivity $\lambda$.

Next, suppose that $\lambda$ is even and $\delta = \Delta$ is odd. In this case, $n$ must be even and $n \geq 2\delta + 4$. Again, we use the modified $G_1$ but we change $V$. Since both $\delta$ and $n - \delta - 2$ are odd, $H_{n-\delta-2,\delta}$ has a vertex, $\frac{n-\delta-3}{2}$, of degree $\delta + 1$. Subtracting $\frac{n-\delta-3}{2}$ from the label of each vertex in $V$, we let $V = \{-1, 0, 1, 2, \ldots, \Delta - \lambda - 2\}$. At this moment, each vertex in $V$ has degree $\delta + 1$ except the degree of 0 is $\delta + 2$. Removing the edges $\{e_{-1}, e_0, e_1, e_2, \ldots, e_{\Delta-\lambda-2}\}$ leaves every vertex with degree $\delta$.

Note that $\Delta - \lambda \geq 2\lambda - \lambda = \lambda$ and so $\Delta - \lambda \geq 3$ if $\lambda \geq 3$. If $\lambda = 2$, then since $\Delta$ is odd and greater than twice $\lambda$, again $\Delta - \lambda \geq 3$. Since $n - \delta - 2 \geq 2\delta + 4 - \delta - 2 \geq \Delta + 2 > \Delta - \lambda$, $V$ is well-defined.

Our final case is for $\lambda$ and $\delta = \Delta$ odd. For $\lambda \geq 3$, we use $G_1$ but here $U = \{0, 1, \ldots, \Delta - \lambda - 1\}$ and $V = \{-1, 0, 1, \ldots, \lambda - 2\}$. If $\lambda = 1$, then we let $G = H_{\delta+2,\delta} \lor H_{n-\delta-2,\delta}$ with an edge between the
two vertices of degree $\delta - 1$. 
Realizability of Larger Cases

Whenever $\kappa \geq 2$, realizability for a graph $G$ can be broken down into the following three cases:

\begin{align*}
(i) \kappa + \Delta &> \lambda + \delta \\
(ii) \kappa + \Delta &= \lambda + \delta \\
(iii) \kappa + \Delta &< \lambda + \delta
\end{align*}

These three cases have several subcases, which will be addressed as they arise. First, consider case (i), $\kappa + \Delta > \lambda + \delta$.

**Theorem:** When $\kappa + \Delta > \lambda + \delta$ and $\lambda = \delta$, $F_{\kappa,\lambda,\delta,\Delta} = \{ n \in \mathbb{N} : n \geq \Delta + 1 \}$.

**Proof.** Let $G_1 = K_{\delta-k+1}$ and $K = K_\kappa$, and consider the graph $G_1 \lor K$, where $\delta - \kappa + 1$ vertices in $G_1$ are adjacent to all $\kappa$ vertices in $K$. So every vertex in $G_1$ has degree $\delta$ and each vertex in $K$ has degree $\kappa$. Now consider $G_2 = K_{\Delta-\lambda}$. Connecting every vertex in $G_2$ to $K$ gives each vertex in $K$ degree $\Delta$ and each vertex in $G_2$ degree $\Delta - \lambda - 1 + \kappa$. Since $\kappa + \Delta > \lambda + \delta$, $\Delta + \kappa - \lambda - 1 \geq \delta$. So $\delta$ is preserved in $G_2$. Since $\lambda = \delta$, every vertex in $G_1 \lor K \lor G_2$ has degree $\geq \delta = \lambda$. Since $\text{deg}_{G_1}(k) = \delta - \kappa + 1$ and $\text{deg}_{G_1}(k) = \Delta - \lambda$ for each vertex $k \in K$, the minimum number of edges that must be removed in order to disconnect a vertex $k \in K$ is $\delta - \kappa + 1 + \kappa - 1 + \Delta - \lambda = \Delta > \lambda$. So $G_1 \lor K \lor G_2$ preserves all parameters $\kappa + \Delta > \lambda + \delta$, $\lambda = \delta$. We know that $n_{\kappa,\lambda,\delta,\Delta}$ under these conditions is no smaller than $\Delta + 1$, since any graph with maximum degree $\Delta$ must have at least $\Delta + 1$ vertices. So $F_{\kappa,\lambda,\delta,\Delta} = \{ n \in \mathbb{N} : n \geq \Delta + 1 \}$ holds. 

\[
\begin{array}{c}
K_{\delta-(\kappa-1)} \quad \text{and} \quad K_\kappa \quad \Delta - \lambda \quad K_{\Delta-\lambda}
\end{array}
\]

\[
F_{\kappa,\lambda,\delta,\Delta} \text{ where } \kappa + \Delta > \lambda + \delta \text{ and } \lambda = \delta:
\]

\[
\begin{array}{c}
K_{\delta-(\kappa-1)} \quad \text{and} \quad K_\kappa \quad \Delta - \lambda \quad H_{n-(\Delta-\lambda),\delta}
\end{array}
\]

**Theorem:** When $\kappa + \Delta > \lambda + \delta$ and $\lambda < \delta$, $F_{\kappa,\lambda,\delta,\Delta} = \{ n \in \mathbb{N} : n \geq \max\{2\delta + 2, \Delta + \delta + \kappa - \lambda + 1\} \}$.

**Proof.** Case 1: $n \geq 2\delta + 2$

Let $G_1 = K_{\delta-k+1}$ and $K = K_\kappa$ and consider the graph $G_1 \lor K$, where all $\delta - \kappa + 1$ vertices in $G_1$ are adjacent to each vertex in $K$. So each vertex in $G_1$ has degree $\delta - \kappa + 1 - 1 + \kappa = \delta$ and each vertex in $K$ has degree $\kappa - 1 + \delta - \kappa + 1 = \delta$. Now consider the graph $G_2 = K_{\delta+1}$, so each vertex has degree $\delta$. Connect each vertex $k \in K$ to a unique vertex $u \in G_2$, so that $\text{deg}(k) = \delta + 1$ for each $k \in K$ and $\text{deg}(u) = \delta + 1$ for $\kappa$ vertices $u \in G_2$. Now, choose one vertex $k_1 \in K$, and connect it to $\lambda - \kappa$ new vertices in $G_2$. We know that there are enough distinct vertices in $G_2$ so
that this is possible because \( \kappa + \lambda - \kappa = \lambda < \delta + 1 = |G_2| \). Now those vertices that are adjacent to \( k_1 \) have degree \( \delta + 1 \leq \Delta \). \( \deg(k_1) = \delta + 1 + \lambda - \kappa \), and since \( \kappa + \Delta > \lambda + \delta \), \( \delta + \lambda - \kappa + 1 \leq \Delta \).

Suppose \( \delta + \lambda - \kappa + 1 = \Delta \). Then connecting \( K \) to \( G_2 \) in this way leaves all vertices with degree \( \geq \delta \). Now consider the graph \( G_1 \lor K \lor G_2 \). Since each component is complete, in order to disconnect a single vertex, you would have to remove \( \geq \delta > \lambda \) edges from it since each vertex has degree \( \geq \delta \). In order to disconnect a component, you would have to remove \( e_{G_1,K} = \kappa(\delta - \kappa + 1) > \lambda \) vertices or \( e_{K,G_2} = \lambda \) edges. So \( \lambda \) is realized. Hence, the graph \( G_1 \lor K \lor G_2 \) realizes all parameters on a minimum of \( 2\delta + 2 \) vertices.

If \( \delta + \lambda - \kappa + 1 < \Delta \), then \( \Delta - \delta - \kappa - \lambda + 1 > 0 \Rightarrow \Delta + \delta + \kappa - \lambda + 1 > 2\delta + 2 \), a contradiction since we are assuming that \( 2\delta + 2 = \max(2\delta + 2, \Delta + \delta + \kappa - \lambda + 1) \).

Case 2: \( n \geq \Delta + \delta + \kappa - \lambda + 1 \)

Let \( G_1 = K_{\delta+1} \) and \( G_2 = K_{\Delta - \lambda + \kappa} \). Now, choose a set of vertices \( L \subset G_2 \) and \( M \subset G_1 \) such that \( |L|=\kappa \) and \( |M|=\lambda \), and connect each vertex \( v \in L \) to a single vertex \( u \in M \). Now, each vertex in \( L \) has degree \( \Delta - \lambda + \kappa \) and \( \kappa \) vertices in \( G_1 \) have degree \( \delta + 1 \leq \Delta \). Now, choose a vertex \( v_1 \in L \) and connect it to the remaining \( \lambda - \kappa \) vertices in \( M \), so that every vertex in \( M \) now has degree \( \delta + 1 \) and \( \deg(v_1) = \Delta - \lambda + \kappa - 1 + 1 + \lambda - \kappa = \Delta \).

Since each component is complete, in order to disconnect a single vertex, you would have to remove \( \geq \delta > \lambda \) edges from it since each vertex has degree \( \geq \delta \). In order to disconnect a component, you would have to remove \( e_{G_1,G_2} = \lambda \) edges. So \( \lambda \) is realized. Hence, the graph \( G_1 \lor G_2 \) realizes all parameters on a minimum of \( \Delta + \delta + \kappa - \lambda + 1 \) vertices.

\[ n_{\kappa, \lambda, \delta, \Delta} \text{ where } \kappa + \Delta > \lambda + \delta \text{ and } \lambda < \delta: \]

**Case 1**

\[ n_{\kappa, \lambda, \delta, \Delta} \text{ where } \kappa + \Delta > \lambda + \delta \text{ and } \lambda < \delta: \]

\[ K_{\delta-(\kappa-1)} \quad \delta - \kappa + 1 \quad K_{\kappa} \quad k_1 \lor \lambda - \kappa + 1 \quad K \setminus \{k_1\} \lor 1 \quad K_{\delta+1} \]

**F$_{\kappa, \lambda, \delta, \Delta}$** where \( \kappa + \Delta > \lambda + \delta \) and \( \lambda < \delta: \)

\[ K_{\delta-(\kappa-1)} \quad \delta - \kappa + 1 \quad K_{\kappa} \quad k_1 \lor \lambda - \kappa + 1 \quad H_{n-\delta-1, \delta} \quad K \setminus \{k_1\} \lor 1 \]

**Case 2**

\[ n_{\kappa, \lambda, \delta, \Delta} \text{ where } \kappa + \Delta > \lambda + \delta \text{ and } \lambda < \delta: \]

\[ K_{\delta+1} \quad \lambda \quad K_{\Delta-\lambda+\kappa} \]

**F$_{\kappa, \lambda, \delta, \Delta}$** where \( \kappa + \Delta > \lambda + \delta \) and \( \lambda < \delta: \)
Case, (ii), \( \kappa + \Delta = \lambda + \delta \).

**Theorem:** When \( \kappa + \Delta = \lambda + \delta \) and \( \lambda = \delta \), \( F_{\kappa, \lambda, \delta, \Delta} = \{n \in \mathbb{N} : n \geq \Delta + 2\} \).
This case has several subcases: \( \Delta + 2 \leq n < \Delta + \kappa + 1 \), \( n = \Delta + \kappa + 1 = 2\delta + 1 \), and \( n \geq \Delta + \kappa + 2 \).

Case 1: \( \Delta + 2 < n < \Delta + \kappa + 1 \)

**Proof.** Let \( L = K_{\delta+1-\kappa}, K = H_{\kappa, \kappa-1-\delta} \) or if \( \kappa \delta \) is odd, then \( K = H_{\kappa, \kappa-1-\delta, \kappa} \), and \( M = H_{\delta, \kappa+1-\delta, \kappa} \) where \( 1 \leq i < \kappa \). Define \( G = (L + M) \vee K \) where \( L + M \) is the disjoint union of \( L \) and \( M \) and every vertex of \( K \) is connected to every vertex of \( L + M \).

Every vertex of \( L + M \) has degree either \( \delta \) or \( \delta_1 \) in \( G \). Note that if \( \delta - \kappa + i \) and \( \delta - \kappa \) are both odd, then \( M \) has a vertex of degree \( \delta - \kappa + 1 \). A vertex \( k \in K \) with \( \text{deg}_K(k) = \kappa - 1 - i \) has \( \text{deg}_G(k) = (\delta + 1 - \kappa) + (\delta - \kappa + i) + (\kappa - 1 - i) = 2\delta - \kappa = \Delta \).

Case 2: \( n = \Delta + \kappa + 1 = 2\delta + 1 \)

**Proof.** Note that since \( \delta \geq \lceil \frac{n}{2} \rceil \), Theorem 3 from the generic theorems section guarantees that \( \lambda = \delta \). By Boesch and Suffel’s conditions for realizability, there is a graph that realizes \((n, \kappa, \lambda, \delta, \Delta)\).

Case 3: \( n \geq \Delta + \kappa + 2 \)

**Proof.** This case actually divides further, into 2 subcases: \( \delta - \kappa = 1 \) and \( \delta - \kappa > 1 \). Since the former has subcases, we will examine the latter first and assume that \( \delta - \kappa > 1 \).

Let \( G = K_{\delta+1} + H_{n-\delta-1, \delta} \) where again this is the disjoint union of the two graphs. We add the following edges between the two components of \( G \): \( \{e_{0,i} : - \leq i < \delta - \kappa\} \cup \{e_{1, \delta - \kappa}, e_{1, \delta - \kappa + 1}\} \cup \{e_{i, \delta - \kappa + 1} : 2 \leq i \leq \kappa - 1\} \). If \( n - \delta - 1 \) and \( \delta \) are both odd, then let vertex \( \delta \) in \( H_{n-\delta-1, \delta} \) have degree \( \delta + 1 \). Since we added \( \lambda = \delta \) edges between the components with \( \kappa \) end-vertices of these edges in \( K_{\delta} \), we have our desired graph.

For our last case, we will find a graph that realizes \((\kappa, \kappa+1, \kappa+1, \kappa+2)\). In cases 1 and 2, we showed that these 4-tuples are realized for \( \Delta + 2 \leq n \leq \Delta + \kappa + 1 \).

If \( n \geq \Delta + \kappa + 2 \), then \( n \geq 2\kappa + 4 \). If \( N_{\kappa} \) is the null graph on \( k \) vertices, then \( K_{2} \vee N_{\kappa} \vee N_{\kappa} \vee K_{2} \) realizes \((\kappa, \kappa+1, \kappa+1, \kappa+2)\). Note that \( N_{\kappa} \vee N_{\kappa} \) is the complete bipartite graph \( K_{\kappa, \kappa} \).

We now use the notation \( H \vee l K \) to indicate that we are adding \( l \) edges between components in \( H + K \) as evenly as possible. To be precise, label the vertices in \( H \) as \( \{0, \ldots, |H| - 1\} \) and the vertices in \( K \) likewise. We add the edge \( e_{0,0} \) first, \( e_{1,1} \) next, etc.
There are two cases, \( \kappa \) even and \( \kappa \) odd. For the even case, let \( G = K_2 \lor H_{\kappa,k-1} \lor 2^k H_{n-k-2,1} \) where

\[
l = \begin{cases} 
\kappa, & \text{if } 2\kappa + 4 < n < 3\kappa + 2; \\
\kappa + 1, & n \geq 3\kappa + 2.
\end{cases}
\]

For the odd case, let \( G = K_2 \lor H_{\kappa,k-2} \lor 2^k H_{n-k-2,1} \) where

\[
l = \begin{cases} 
\kappa, & \text{if } 2\kappa + 4 < n < 3\kappa + 1; \\
\kappa + 1, & n \geq 3\kappa + 1.
\end{cases}
\]

Since some of the Haray graphs may have a vertex of one larger degree than the other vertices, we use that vertex last when adding edges. It is not difficult to check that this construction ensures that there are enough edges between components to preserve \( \lambda \) and that the degree of each vertex is either \( \kappa + 1 \) or \( \kappa + 2 \).

\[n_{\kappa,\lambda,\delta,\Delta} \text{ where } \kappa + \Delta = \lambda + \delta \text{ and } \lambda = \delta:\]

**Case 1:** \( \Delta + 2 \leq n < \Delta + \kappa + 1 \)

**Case 3(i):** \( n \geq \Delta + \kappa + 2 \) where \( \delta - \kappa > 1 \)

**Case 3(ii):** \( n \geq \Delta + \kappa + 2 \) where \( \delta - \kappa = 1 \)

**Theorem:** When \( \kappa + \Delta = \lambda + \delta \) and \( \lambda < \delta \), \( F_{\kappa,\lambda,\delta,\Delta} = \{ n \in \mathbb{N} : n \geq 2\delta + 2 \} \).

**Proof.** Let \( G_1 = K_{\delta-\kappa+1} \) and \( K = K_{\kappa} \) and consider the graph \( G_1 \lor K \), where each vertex in \( G_1 \) is adjacent to each vertex in \( K \). So \( \text{deg}(u) = \delta - \kappa + 1 - 1 + \kappa = \delta \) for each \( u \in G_1 \) and \( \text{deg}(k) = \kappa - 1 + \delta - \kappa + 1 = \delta \) for each \( k \in K \). Now consider the graph \( G_2 = K_{\delta+1} \), so each vertex in \( G_2 \) has degree \( \delta \). Connect all vertices \( k \in K \) with one edge to a distinct vertex \( u \in G_2 \). So now \( \text{deg}(k) = \delta + 1 \) for each \( k \in K \), and \( \delta \leq \text{deg}(u) \leq \delta + 1 \) for each \( u \in G_2 \). Now, choose one vertex \( k_1 \in K \) and connect it to vertices with degree \( \delta \) in \( G_2 \) with \( \lambda - \kappa - 1 \) edges. Since \( \kappa + \Delta = \lambda + \delta \), \( \Delta - \delta = \lambda - \kappa \), so there are enough vertices left in \( G_2 \) for \( k_1 \) to connect to. Now all vertices in \( G_2 \), except one, have degree \( \delta + 1 \leq \Delta \) and \( \text{deg}(k_1) = \delta + \lambda - \kappa - 1 = \Delta \). Right now, \( e_{K,G_2} = \lambda - \kappa - 1 + 1 + \kappa - 1 = \lambda - 1 \), so choose a vertex \( k_2 \in K \setminus \{ k_1 \} \) and connect it to the remaining vertex in \( G_2 \) that has degree \( \delta \). So \( \text{deg}(k_2) = \delta + 1 + \lambda - \kappa - 1 = \delta + \lambda - \kappa = \Delta \) and \( \text{deg}(u) = \delta + 1 \) for each vertex \( u \in G_2 \).

Since \( G_1, K, \) and \( G_2 \) are all complete graphs, the only way to disconnect a single vertex \( v \in G_1 \lor K \lor G_2 \) is to remove \( \text{deg}(v) \geq \delta > \lambda \) edges. In order to disconnect a component, a minimum of \( \lambda \) edges must be removed, which can be done by removing the edges between \( K \) and \( G_2 \). Thus,
G₁ ∨ K ∨ G₂ preserves all parameters. By Theorem 1 in the λ < δ section, nₖ,λ,δ,∆ = 2δ + 2 is the smallest possible graph with these parameters.

nₖ,λ,δ,∆ where \( \kappa + \Delta = \lambda + \delta \) and \( \lambda < \delta \):

\[
\begin{align*}
K_{\delta-(\kappa-1)} & \quad \delta - \kappa + 1 \\
K_{\kappa} & \quad \left\{ k_1, k_2 \right\} \lor \lambda - \kappa \\
K_{\delta+1} & \quad K - \left\{ k_1, k_2 \right\} \lor 1
\end{align*}
\]

Fₖ,λ,δ,∆ where \( \kappa + \Delta > \lambda + \delta \) and \( \lambda < \delta \):

\[
\begin{align*}
K_{\delta-(\kappa-1)} & \quad \delta - \kappa + 1 \\
K_{\kappa} & \quad \left\{ k_1, k_2 \right\} \lor \lambda - \kappa \\
H_{n-\delta-1,\Delta-1} & \quad K - \{ k_1, k_2 \} \lor 1
\end{align*}
\]

**Conclusion**

It is well-known that some graph-theoretic extremal questions play an important role in communication network vulnerability. Questions concerning the realizability of graphs are generalizations of these extremal problems [5]. The \((\kappa, \lambda, \delta, \Delta)\) realizability theorems in this paper solve several extremal problems. We have theorems defining realizability for graphs where \( \kappa = 0 \),

**Theorem:** If \( \kappa = 0 \) then,

\[
F_{\kappa,\lambda,\delta,\Delta} = \{ n \in \mathbb{N} : n \geq \delta + \Delta + 2 \}
\]

unless \( \delta \Delta = 0 \), in which case \( F_{0,0,0,0} = \mathbb{N} \) or \( \delta \Delta \) is odd and \( \delta = \Delta \), in which case \( F_{0,0,\delta,\Delta} = \{ n \in \mathbb{E} : n \geq 2\delta + 2 \} \).

Hence \( n_{\kappa,\lambda,\delta,\Delta} = \delta + \Delta + 2 \) unless \( \delta \Delta = 0 \) in which case \( n_{0,0,0,0} = 1 \);

and \( \kappa = 1 \),

**Theorem:** If \( \kappa = 1 \), then

\[
F_{1,\lambda,\lambda,\Delta} = \{ n \in \mathbb{N} : n \geq \Delta + 1 \}
\]

unless \( \Delta = 1 \), in which case \( F_{1,1,1,1} = 2 \). For \( \lambda < \delta < \Delta \) and \( \lambda + \delta \leq \Delta \),

\[
F_{1,\lambda,\delta,\Delta} = \{ n \in \mathbb{N} : n \geq \Delta + \delta - \lambda + 2 \}.
\]

For \( \lambda < \delta < \Delta \) and \( \lambda + \delta > \Delta \) or for \( \delta = \Delta \) and \( \lambda \) all even,

\[
F_{1,\lambda,\delta,\Delta} = \{ n \in \mathbb{N} : n \geq 2\delta + 3 \}
\]

For \( \delta = \Delta \) odd,

\[
F_{1,\lambda,\delta,\Delta} = \{ n \in \mathbb{E} : n \geq 2\delta + 4 \}.
\]

Otherwise (for example, \( 2\lambda > \Delta \))

\[
F_{1,\lambda,\delta,\Delta} = \emptyset.
\]

For \( \kappa \geq 2 \), we have theorems covering several subcases. When \( \kappa + \Delta > \lambda + \delta \),

**Theorem:** If \( \lambda = \delta \), then \( F_{\kappa,\lambda,\delta,\Delta} = \{ n \in \mathbb{N} : n \geq \Delta + 1 \} \) and

**Theorem:** If \( \lambda < \delta \), then \( F_{\kappa,\lambda,\delta,\Delta} = \{ n \in \mathbb{N} : n \geq \max(2\delta + 2, \Delta + \delta + \kappa - \lambda + 1) \} \).
When $\kappa + \Delta = \lambda + \delta$, we have

**Theorem:** If $\lambda = \delta$, then $F_{\kappa,\lambda,\delta,\Delta} = \{n \in \mathbb{N} : n \geq \Delta + 2\}$ and,

**Theorem:** If $\lambda < \delta$, then $F_{\kappa,\lambda,\delta,\Delta} = \{n \in \mathbb{N} : n \geq 2\delta + 2\}$.

There is a third case still left to investigate: when $\kappa + \Delta < \lambda + \delta$. There are several subcases within this case and the relationship between the parameters is very subtle. At this time, we are unable to find a pattern to generalize realizability, but we have collected a lot of data on this case.

**References**


