Realizability of $n-$Vertex Graphs with Prescribed Vertex Connectivity, Edge Connectivity, Minimum Degree, and Maximum Degree

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1 Introduction

Our work continues a long tradition in graph theory studying the relationship between vertex connectivity, edge connectivity, and minimum degree. Whitney [14] showed that for any graph the minimum degree is at least the edge connectivity which is at least the vertex connectivity and Chartrand and Harary [5] showed that given any three positive integers in non-decreasing order, there is a graph with vertex connectivity the smallest of the numbers, minimum degree the largest of them, and edge connectivity the middle of the numbers. Before describing our work, we present some definitions.

Definition 1.1. For a connected, but not complete graph G, we define *vertex connectivity* $\kappa(G)$ to be the smallest number of vertices whose removal results in a disconnected graph. If G is not connected, then $\kappa(G) = 0$. If G is a complete graph K_n, then $\kappa(G) = n - 1$.

Definition 1.2. For a connected graph, G, we define *edge connectivity* $\lambda(G)$ to be the smallest number of edges whose removal results in a disconnected graph. If G is not connected or if $G = K_1$, then $\lambda(G) = 0$. We also define a *bond* to be a minimal set of edges needed to disconnect a connected graph.

Definition 1.3. The *degree* of a vertex u in a graph G is the number of edges incident to u. An edge is *incident* to a vertex u if u is one of the endpoints of the edge.

Notation 1.4. *We will denote the degree of a vertex u by $\rho(u)$.*

Definition 1.5. For a graph G, we define *minimum degree* $\delta(G)$ to be the least number of edges incident to any vertex of G. For a graph G, we define *maximum degree* $\Delta(G)$ to be the greatest number of edges incident to any vertex of G.

Our research studies only simple graphs with no loops. We now define these.

Definition 1.6. A *loop* is an edge whose endpoints are the same vertex.

Definition 1.7. A *simple graph* is a graph with no multiple edges and no loops.

With these definitions in mind, we define the size of a graph.

Definition 1.8. The *size* of a graph G is the number of vertices in G. If a graph G has n vertices, then the size of G is n.

Next we list these useful and well-known facts concerning the parameters we are studying.

Fact 1.9 (Whitney [14])**.** *For any graph G, we have $\kappa(G) \leq \lambda(G) \leq \delta(G) \leq \Delta(G)$.*

Fact 1.10 (Chartrand and Harary [5])**.** *Given three positive integers κ, λ, and δ with $\kappa \leq \lambda \leq \delta$, there is a graph G with $\kappa(G) = \kappa$, $\lambda(G) = \lambda$, and $\delta(G) = \delta$.*

Fact 1.11 (West [13])**.** *For a connected graph with $\lambda(G) < \delta(G)$ deleting a bond of size λ leaves two components, each with at least $\delta + 1$ vertices.*

Fact 1.12 (Chartrand and Harary [5])**.** *Deleting an edge from a graph decreases its vertex connectivity by at most 1.*

These definitions and results raise further questions about other possible restrictions that can be put on the parameters. The particular question we ask is: given prescribed vertex connectivity κ, edge connectivity λ, minimum degree δ, and maximum degree Δ, what is the minimum size n of a graph that realizes $(\kappa, \lambda, \delta, \Delta, n)$? We say that the parameters $(\kappa, \lambda, \delta, \Delta, n)$ are realizable if there exists a graph G such that the size of G is n and $\kappa(G) = \kappa$, $\lambda(G) = \lambda$, $\delta(G) = \delta$, and $\Delta(G) = \Delta$. When we consider prescribed vertex connectivity, edge connectivity, minimum degree, and maximum degree we do so by placing restrictions on how the parameters can relate to each other given previous results. Two basic facts are as follows. The first was proven by West [13].

Fact 1.13 (West [13]). *If $\lambda < \delta$, then $n \geq 2\delta + 2$.*

Observation 1.14. *If $\lambda = \delta$, then $n \geq \Delta + 1$.*

Notation 1.15. *Unless otherwise noted, if L and M are graphs then $L \cup M$ is the graph with vertex set $V(L) \cup V(M)$ and edge set $E(L) \cup E(M)$ where $V(L)$ and $V(M)$ are the vertices of L and M respectively and $E(L)$ and $E(M)$ are the edges of L and M respectively.*

Notation 1.16. *The number of vertices or a set of edges in a set S will be denoted n_S if S is a set of vertices or m_S if S is a set of edges. If deleting a bond S from G disconnects G, then the graph $G - S$ is $L \cup M$ with $n_L \leq n_M$ where L is the component of the disconnected graph $G - S$ with the fewer vertices and M is the component of the disconnected graph $G - S$ with the most vertices.*

Definition 1.17. A *component* of a graph G is a subset of vertices and edges of G in which any two vertices are connected by a path and no vertex in the subset is connected to a vertex not in the subset. A *path* in a graph G is a sequence of edges in G that connect a sequence of vertices in G. For our purposes, these vertices will be distinct.

Notation 1.18. *Let G be a graph and let S and T be any disjoint sets of vertices of G. We denote by [S,T] all edges incident to a vertex S and a vertex in T. If $S = \{u\}$, then this set of edges is denoted by [u, T].*

Notation 1.19. *If κ, λ, δ, and Δ are non-negative integers, then we define $F(\kappa, \lambda, \delta, \Delta)$ to be the set of all integers n such that there is a graph G on n vertices with $\kappa(G) = \kappa$, $\lambda(G) = \lambda$, $\delta(G) = \delta$, and $\Delta(G) = \Delta$. Any such graph is said to realize the 4-tuple $(\kappa, \lambda, \delta, \Delta)$. If $F(\kappa, \lambda, \delta, \Delta)$ is non-empty, then we let $f(\kappa, \lambda, \delta, \Delta)$ be the smallest element in $F(\kappa, \lambda, \delta, \Delta)$.*

Many of our results involve the use of Harary and complete graphs, which are graphs that are well-defined and have many previously examined properties.

Definition 1.20. A *complete* graph G of $\{u_0, u_1, \ldots, u_{n-1}\}$ vertices is a simple graph such that for all u_i and u_j where $i \neq j$ and $0 \leq i, j \leq n - 1$ there is an edge that connects u_i and u_j.

Definition 1.21. The vertices of a *Harary* graph, $H_{n,k}$ are $0, 1, 2, \ldots, n-1$. If k even, then vertex i adjacent to vertex $i+j$ and i is adjacent to vertex $i-j$ for $0 < j \leq k/2$. If k is odd and n is even, then the Harary graph $H_{n,k}$ is $H_{n,k-1}$ with added edges joining i and $i+n/2$. If k and n are both odd, then the Harary graph $H_{n,k}$ is $H_{n,k-1}$ with the added edges joining i and $i+(n-1)/2$. Note that vertex $(n-1)/2$ has degree $k+1$.

What is of interest to us is that in that paper he constructed graphs that are now known as the Harary graphs. Harary [11] showed that the minimum number of edges in a κ-connected graph on n vertices is $\lceil \frac{kn}{2} \rceil$. For $n > k \geq 0$ and nk even, a Harary graph, $H_{n,k}$, is a regular graph on n vertices of degree k with vertex and edge connectivity k. If n and k are both odd, then $H_{n,k}$ has minimal degree and vertex and edge connectivity k but has one vertex of degree $k+1$.

Note that if $k > 1$, then the Harary graph $H_{n,k}$ is hamiltonian. Also, if nk is odd, then there is a vertex of degree $k+1$ and removing an edge from that vertex results in a graph with minimum degree, vertex connectivity, and edge connectivity all $k-1$ and all but one vertex has degree k. We denote such a graph by $H^-_{n,k}$.

For $k = 0$, the Harary graph $H_{n,0}$ is the null graph on n vertices. For $k = 1$ and n even, $H_{n,1}$ is the union of $\frac{n}{2}$ copies of K_2. For n odd, $H_{n,1}$ is the union of $\frac{n-3}{2}$ copies of K_2 and one copy of P_3, the path on three vertices.

Given these facts we are able to put restrictions on the parameters and ask whether or not $(\kappa, \lambda, \delta, \Delta)$ is realizable for all n described by these facts, and if, so what is a graph that realizes the parameters? We specifically examine the case when $\kappa + \Delta < \lambda + \delta$. The reason we examine this case is also what makes our research different than that of Boesch [1], Boesch and Suffell [2, 3, 4], and DiMarco [6, 7, 8, 9, 10]: we view n as a function of κ, λ, δ, and Δ. Dymacek and Hardnett have described the n for which there is a graph with n vertices that realizes $(\kappa, \lambda, \delta, \Delta)$ when $\kappa + \Delta \geq \lambda + \Delta$ or $\kappa = 0$ or $\kappa = 1$. For this reason, we consider $\kappa + \Delta < \lambda + \delta$ in particular. The $\kappa + \Delta < \lambda + \delta$ case is exceptionally restrictive because there is an upper bound constraint in addition to the lower bound constraint on Δ. In the case when $\kappa + \Delta \geq \lambda + \delta$, Δ could be made large to satisfy different sizes of n given other restrictions on κ, λ, and δ. However, when $\kappa + \Delta < \lambda + \delta$, we cannot do this. Because of the restrictiveness of $\kappa + \Delta < \lambda + \delta$, we state our results using several sub-cases of conditions for $(\kappa, \lambda, \delta, \Delta)$. Notice that in some subcases $(\kappa, \lambda, \delta, \Delta)$ is not realizable for all n given by Fact 1.13 and

Fact 1.14, and we prove this.

This question has been previously examined. Whether or not a graph is realizable has been asked by both Boesch and Suffel and DiMarco in terms of connectivity parameters, both vertex connectivity and edge connectivity, and in terms of size, defined both by number of vertices and number of edges. The most immediate way our research differs from previous research is that we consider vertex connectivity, edge connectivity, minimum degree, and maximum degree together. We must first consider this result.

Theorem 1.22. *If G is regular and $\lambda < \delta$, then λ has the same parity as the degree of G times the size of either component after a bond is removed from G.*

Proof. Let C be one of the two components of G after removing a bond of G. Let n_C be the number of vertices in a component and m_C be the number of edges. Since each edge of the bond is incident to a vertex of C, $n_c \cdot \delta = 2m_c + \lambda$. Thus $n_c \cdot \delta - \lambda = 2m_c$. ∎

This has two immediate corollaries regarding edge connectivity and minimum degree.

Definition 1.23. A graph G is *regular* if minimum degree and maximum degree are equal for all vertices of G.

Corollary 1.24. *If G is regular and $\lambda < \delta$ where δ is even, then λ is even.*

Corollary 1.25. *For $\lambda < \delta$, the sequence $(\kappa, \lambda, \delta, \delta)$ is not realizable if λ is odd and δ is even.*

2 $\kappa < \lambda < \delta < \Delta$

The first restriction on $(\kappa, \lambda, \delta, \Delta)$ that we consider beyond $\kappa + \Delta < \lambda + \delta$ is the strict inequality case when $\kappa < \lambda < \delta < \Delta$. From previous research we have that if $\lambda < \delta$, then $n \geq 2\delta + 2$. Note that choosing κ, λ, and δ restricts the options for Δ such that, for small choices of parameters and small n, we can construct specific graphs to realize $(\kappa, \lambda, \delta, \Delta, n)$. In particular, we are interested in how to construct the graphs that realize the parameters when the difference between minimum and maximum degree is small versus when the difference between minimum and maximum degree is large.

Our results for the case when $\kappa < \lambda < \delta < \Delta$ are defined by this difference. We specifically separate the 4-tuples which satisfy $\Delta - \delta \geq \frac{\lambda}{\kappa} - \kappa + 1$ from those which satisfy $\Delta - \delta < \frac{\lambda}{\kappa} - \kappa + 1$. For both cases we consider the case when κ is even as well as the case when κ is odd. The reason for this lies in how the graph is constructed to satisfy vertex connectivity and edge connectivity. The graph for the case when $\kappa < \lambda < \delta < \Delta$ has two blocks, which we will denote L and M, and we divide the κ cut vertices as evenly as possible amongst them, which is exactly half when κ is even. In the case when κ is odd, we produce a method for reducing the construction to that of the even case.

Main Theorem I: $\Delta - \delta \geq \frac{\lambda}{\kappa} - \kappa + 1$.

Theorem 2.1. *Given $\kappa < \lambda < \delta < \Delta$ with $\kappa + \Delta < \lambda + \delta$ and $\Delta - \delta \geq \frac{\lambda}{\kappa} - \kappa + 1$, then $(\kappa, \lambda, \delta, \Delta)$ is realizable for $n \geq 2\delta + 2$.*

Observation 2.2. *Given these hypotheses, $\delta > \kappa + 2$.*

Proof. Since $\lambda < \delta$ we know that $n \geq 2\delta + 2$. Thus we only need to construct graphs with these parameters for each $n \geq 2\delta + 2$.

Consider $K_{\delta+1}$ with vertices $\{u_0, \ldots u_{\kappa-1}, u_\kappa, \ldots, u_\delta\}$. Let $L = K_{\delta+1}$ where we remove the edges of $H_{\kappa,1}$ from the U with vertices $\{u_0, \ldots, u_{\kappa-1}\}$. If κ is odd, then we remove two edges incident to $u_{\kappa-1}$. Let $M = H_{n-\delta-1,\delta}$ with vertices $\{v_0, \ldots, v_{n-\delta-2}\}$ where if both n and δ are odd, then $\rho_M(v_{n-\delta-2}) = \delta + 1$. Note that $n - \delta - 1 > \delta$ or $n > 2\delta + 1$. Therefore M is well-defined and has the correct number of vertices. We define the graph G of interest to be

$$L \cup M$$

with λ additional edges joining L and M. These edges will be described below but we first need to define some parameters.

Note that

$$\rho_L(u_i) = \begin{cases} \delta, & \text{for } \kappa \leq i \leq \delta \\ \delta - 1, & \text{for } 0 \leq i < \kappa \text{ and } \kappa \text{ even} \\ \delta - 2, & \text{for } i = \kappa - 1 \text{ and } \kappa \text{ odd.} \end{cases} \tag{1}$$

Let

$$d = \lambda + \delta - \kappa - \Delta. \tag{2}$$

Thus when $d = 1$, Δ is as large as possible given our hypotheses. For an upper bound, $\frac{\lambda}{\kappa} - \kappa + 1 \leq \Delta - \delta = \lambda - \kappa - d$ and so $\frac{\lambda}{\kappa} + 1 \leq \lambda - d$ or $d \leq \lambda(1 - \frac{1}{\kappa}) - 1$. Since $\Delta - \delta \geq 1$, we also have $d \leq \lambda - \kappa - 1$. If $\lambda \leq \kappa^2$, then $\frac{\lambda}{\kappa} - \kappa + 1 \leq 1$ and so $\Delta - \delta \geq 1$ and in this case $1 \leq d \leq \lambda - \kappa - 1$. If $\lambda(1 - \frac{1}{\kappa}) - 1 > \lambda - \kappa - 1$, then $\kappa^2 > \lambda$. Thus

$$\begin{cases} 1 \leq d \leq \lambda - \kappa - 1, & \text{if } \lambda \leq \kappa^2 \\ 1 \leq d \leq \lambda(1 - \frac{1}{\kappa}) - 1, & \text{if } \lambda > \kappa^2. \end{cases} \tag{3}$$

Finally, let

$$e_d = \left\lceil \frac{d}{\kappa - 1} \right\rceil. \tag{4}$$

Let

$$\ell = \left\lfloor \frac{(\kappa - 1)(\lambda - \kappa)}{\kappa} \right\rfloor = \begin{cases} \lambda - \kappa + 1 - \frac{\lambda}{\kappa}, & \text{if } \kappa \mid \lambda \\ \lambda - \kappa - \left\lfloor \frac{\lambda}{\kappa} \right\rfloor, & \text{otherwise .} \end{cases} \tag{5}$$

The edges that must be added between L and M to produce the desired graph G depend upon whether $d \leq \ell$ or $d > \ell$. Let

$$\alpha_d = (\kappa - 1)e_d - d \qquad \text{and} \qquad \beta_d = (\kappa - 1)(1 - e_d) + d. \tag{6}$$

Note that if $\kappa = 2$, then d is never larger than ℓ. (Since if $\kappa = 2$, then $\lambda \geq 4$. If $\lambda = 4$, then $d = 1 = \ell$ and if $\lambda > 4$, then $d \leq \frac{\lambda}{2} - 1$ and $\ell = \frac{\lambda}{2} - 1$ if λ is even and $\ell = \frac{\lambda - 1}{2} - 1$ if λ is odd.) Also, $e_d = d$ and so $\alpha_d = 0$ and $\beta_d = 1$. We summarized these facts in:

Proposition 2.3. *If $\kappa = 2$, then $d \leq \ell$, $e_d = d$, $\alpha_d = 0$, and $\beta_d = 1$.*

Case I. $d \leq \ell$

In this case we add the edges

$$\big\{\{u_0, v_j\} : 0 \leq j \leq \lambda - \kappa - d\big\} \tag{7}$$

so that

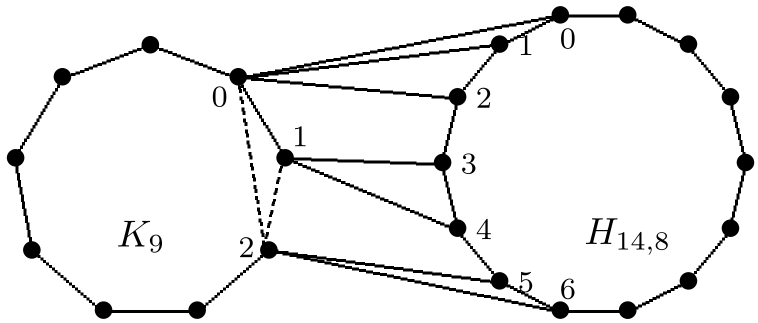

Figure 1: $\kappa = 3$, $\lambda = 7$, $\delta = 8$, $\Delta = 10$, $n = 23$

$$\rho_G(u_0) = \rho_L(u_0) + \rho_M(u_0) = (\delta - 1) + (\lambda - \kappa - d + 1) = \Delta. \tag{8}$$

We also add edges to have α_d vertices in L whose degree in M is e_d and β_d vertices in L whose degree in M is $e_d + 1$. Note that α_d may be 0. The other edges that must be added are

$$\{u_{1+i}, v_{\lambda-\kappa-d+j+ie_d}\} : 0 \le i < \alpha_d,\ \ 1 \le j \le e_d \qquad \text{and}$$

$$\{u_{\alpha_d+i}, v_{\lambda-\kappa-d+\alpha_d e_d+(i-1)(e_d+1)+j}\} : 1 \le i \le \beta_d,\ \ 1 \le j \le e_d + 1\ \ . \tag{9}$$

Proposition 2.4. *For $d \le \ell$, we have $\alpha_d + \beta_d = \kappa - 1$.*

Proof. From (7),

$$\alpha_d + \beta_d = \big((\kappa - 1)e_d - d\big) + \big((\kappa - 1)(1 - e_d) + d\big) = \kappa - 1.$$

■

For an example of this construction, see Figure 1 where the dashed edges are the $H_{3,1}$ that was removed from the K_9. For clarity, not all the edges in the K_9 or the $H_{14,8}$ are displayed. This graph of 23 vertices realizes the sequence $(3, 7, 8, 10)$.

Proposition 2.5. *For $d \le \ell$, we have for all $\kappa \ge 2$ that $\beta_d = 0$ and for $\kappa = 2$ that $\alpha_d = 0$ and $\beta_d = 1$.*

Proof. The first statement is found in Proposition 7. To show the last statement, let $d = (\kappa - 1)q + r$ where q and r are integers and $0 \leq r < \kappa - 1$. Thus

$$e_d = \frac{d}{\kappa - 1} = \frac{(\kappa - 1)q + r}{\kappa - 1} = \begin{array}{ll} q, & \text{if } r = 0; \\ q + 1, & \text{if } r > 0. \end{array}$$

Thus

$$\kappa(1 - e_d) = \begin{array}{l} -(\kappa - 1)q + (\kappa - 1) \\ -(\kappa - 1)q \end{array} = \begin{array}{ll} -d + \kappa - 1, & r = 0; \\ -d + r, & r > 0. \end{array}$$

Therefore, $\begin{array}{ll} \beta_d = d + (\kappa - 1)(1 - e_d) = \kappa - 1, & r = 0; \\ r, & r > 0. \end{array}$

In both cases, $\beta_d > 0$. ■

Proposition 2.6. *If κ is odd, then $\rho(u_{\kappa-1} \geq \delta$.*

Proof. Since $L = K_{\delta-1}$, there are δ edges incident to $u_{\kappa-1}$. If κ is odd, then we remove two edges from $u_{\kappa-1}$. Since $\beta_d \geq 1$, we are adding $e_d + 1$ edges to $u_{\kappa-1}$ and since $e_d \geq 1$, we have added back at least two edges to $u_{\kappa-1}$. ■

Proposition 2.7. *There are λ edges joining L to M.*

Proof. There are $\lambda - \kappa - d + 1$ edges from u_0 to M. There are α_d vertices with e_d edges from L to M and β_d vertices with $e_d + 1$ edges. So we have

$$\begin{aligned} \rho_M(L) &= (\lambda - \kappa - d + 1) + \alpha_d \cdot e_d + \beta_d(e_d + 1) \\ &= \lambda - (\kappa - 1) - d + (\alpha_d + \beta_d)e_d + \beta_d \\ &= \lambda - d - (\kappa - 1) + (\kappa - 1)e_d + (\kappa - 1)(1 - e_d) + d \\ &= \lambda. \end{aligned}$$

■

We now must show that the minimum and maximum degrees of G are δ and Δ respectively.

Proposition 2.8. *For $d \leq \ell$, we have $\delta(G) = \delta$ and $\Delta(G) = \Delta$.*

Proof. All vertices in M have degree δ except vertex $v_{n-\delta-2}$, which has degree $\delta+1$ if n and δ are both odd. The vertices $\{v_0, \ldots, v_{\lambda-1}\}$ in M each had one edge added and so their degrees are $\delta+1$. Since $\lambda < \delta$, $v_{n-\delta-2}$ does not have an edge added to it and so its degree remains $\delta+1$. Since $\Delta > \delta$, all the vertices of M have the correct range of degrees in G.

By (8), $\rho_G(u_0) = \Delta$. By (2), each vertex in L indexed greater than or equal to κ has degree δ. Since $e_d = \lceil \frac{d}{\kappa-1} \rceil \geq 1$, all the other vertices have at least one edge added. Since $\beta_d > 0$, vertex $u_{\kappa-1}$ has at least two edges added. Thus all vertices in L have degree at least δ in G.

All we now need to show is that none of the vertices in $\{u_1, \ldots u_{\kappa-1}\}$ have degree larger than Δ in G. For $0 < i < \kappa$,

$$\Delta - \rho_G(u_i) = \Delta - \rho_L(u_i) - \rho_M(u_i).$$

Note that $\rho_L(u_i) = \delta - 1$ unless κ is odd and $i = \kappa - 1$, in which case $\rho_L(u_i) = \delta - 2$. (In this case, $\rho_M(u_{\kappa-1}) = e_d + 1$.) Since $\Delta = \lambda + \delta - \kappa - d$, we have $\Delta - \delta = (\lambda - \kappa) - d$. Also, $\rho_M(u_i)$ is either e_d or $e_d + 1$. Hence,

$$\Delta - \rho_G(u_i) = \lambda - \kappa + \epsilon - (d + e_d)$$

where $\epsilon = 0$ unless (i) $\rho_M(u_i) = e_d$, (ii) $\kappa = 2$, or (iii) $i = \kappa - 1$ and κ is odd. In these cases, $\epsilon = 1$. Therefore to show that none of the u_i have degree larger than Δ, we just need to show that

$$\lambda - \kappa \geq d + e_d \geq \ell + e_\ell. \tag{10}$$

Since $e_d = \lceil \frac{d}{\kappa-1} \rceil$, we note that $d + e_d$ is strictly increasing. Therefore we need only to consider $\ell + e_\ell$ for $\kappa > 2$. Using (3) when $\kappa = 2$, we have $e_d = d$ and $\epsilon = 1$, and so we want $\lambda - \kappa + 1 \geq 2d$ or $\lambda - 1 \geq 2d$. If $\lambda = 4$, then $d = 1$ and since $3 > 2$, our result holds. If $\lambda > 4$, then $d \leq \frac{\lambda}{2} - 1$. Hence $2d \leq 2(\frac{\lambda}{2} - 1) = \lambda - 2 < \lambda - 1$ and once again our result holds.

Next, to complete the proof of the proposition, we prove the following

Lemma 2.9. *Under our hypotheses and for $\kappa > 2$,*

$$e_\ell + \ell = \begin{array}{lll} \lambda - \kappa - 1 & \textit{if } \lambda \equiv 1 & \pmod{\kappa}; \\ \lambda - \kappa & \textit{otherwise.} & \end{array}$$

Proof. Let $\lambda = \kappa q + r$ where q and r are integers with $0 \leq r < \kappa$. There are three cases: $r = 0$, $r = 1$, and $r > 1$. Using (4) and (5) we have for $r = 0$,

$$
\begin{aligned}
e_\ell + \ell &= \left\lceil \frac{\lambda - \kappa + 1 - \frac{\lambda}{\kappa}}{\kappa - 1} \right\rceil + \lambda - \kappa + 1 - \frac{\lambda}{\kappa} \\
&= \left\lceil \frac{\lambda \;\; 1 - \frac{1}{\kappa} \;\; - (\kappa - 1)}{\kappa - 1} \right\rceil + \lambda - \kappa + 1 - \frac{\lambda}{\kappa} \\
&= \;\; \frac{\lambda}{\kappa} - 1 \;\; + \lambda - \kappa + 1 - \frac{\lambda}{\kappa} \\
&= \lambda - \kappa.
\end{aligned}
$$

Again using (4) and (5),

$$
\begin{aligned}
e_\ell + \ell &= \left\lceil \frac{\lambda - \kappa - \left\lfloor \frac{\lambda}{\kappa} \right\rfloor}{\kappa - 1} \right\rceil + \lambda - \kappa - \left\lfloor \frac{\lambda}{\kappa} \right\rfloor \\
&= \;\; \frac{\kappa q + r - \kappa - q}{\kappa - 1} \;\; + \lambda - \kappa - q \\
&= \;\; \frac{(\kappa - 1)q - (\kappa - r)}{\kappa - 1} \;\; + \lambda - \kappa - q \\
&= \;\; q - \frac{\kappa - r}{\kappa - 1} \;\; + \lambda - \kappa - q \\
&= \;\; \begin{matrix} \lambda - \kappa - 1 & \text{if } r = 1 \\ \lambda - \kappa & \text{if } r > 1. \end{matrix}
\end{aligned}
$$

Therefore using (10), we have proven our claim both for this Lemma and for our Proposition. ■

We now show that $\kappa(G) = \kappa$.

Proposition 2.10. *Under our current assumptions, $\kappa(G) = \kappa$.*

Proof. Consider U with vertices $\{u_0, \ldots, u_{\kappa-1}\}$. Since $G - U$ consists of two disjoint graphs, $L - U$ and M, we have that $\kappa(G) \leq \kappa$.

Note that since $M = H_{n-1-\delta,\delta}$ and $\kappa < \delta$, removing κ or fewer vertices from M leaves it connected. We now will show the same for L.

For $\kappa = 2$ we have removed one edge from $K_{\delta+1}$ to construct G and for $\kappa = 3$ we have removed two edges. Using Fact 1.10 and $\kappa(K_{\delta+1}) = \delta$, we have in these two cases that $\kappa(L) \geq \delta - 2 \geq \kappa$ for L. We now consider $\kappa > 3$.

Let u_i and u_j be vertices in L. In $K_{\delta+1}$, there are δ vertex-disjoint paths from u_i to u_j: $u_i \to u_j$ and $u_i \to u_k \to u_j$ for $0 \leq k \leq \delta$ but $k = i$ and $k = j$. In L we have removed at most one edge from each vertex in U except that we may have removed two edges from $u_{\kappa-1}$. Hence there are at least $\delta - 3 \geq \kappa$ vertex-disjoint paths in L from u_i to u_j. Therefore $\kappa(L) \geq \kappa$.

Removing fewer than κ vertices from G leaves both L and M connected and leaves at least one edge from L to M. Whence $\kappa(G) \geq \kappa$ and therefore, $\kappa(G) = \kappa$. ■

Proposition 2.11. *Under our current assumptions,* $\lambda(G) = \lambda$.

Proof. We constructed G so that removing the λ edges joining L and M disconnects G. Thus $\lambda(G) \leq \lambda$.

Since $\lambda < \delta$, removing λ edges from M leaves M connected. Also by Fact 1.11, removing a bond from G leaves two components, each with at least $\delta + 1$ vertices. Thus if any vertex of L is in the component containing M, the other component must have fewer than $\delta + 1$ vertices, a contradiction. Therefore removing a bond from G leaves two components, L and M and so $\lambda(G) = \lambda$.

■

Case II. $d > \ell$.

This case is similar to Case I in that we use basically the same construction but we change L. Here we let M be exactly the same as in Case I and let $L = K_{\delta+1} - H_{\kappa,d-l+1}$ where the Harary graph is again on the vertices of U and if κ is odd and if d and ℓ have the same parity, then let $u_{\kappa-1}$ be the vertex with the extra edge removed. Note that from Proposition 2.3, we must have $\kappa > 2$.

We also define ℓ as in (5) and α_ℓ and β_ℓ as in (6). The edges we add to join L and M are the same as if $d = \ell$. See equations (7) and (9). The first question we must answer is whether L is defined. Is $\kappa > d - \ell + 1$?

Proposition 2.12. *For the hypotheses in this case, L is well-defined and* $\kappa > d - \ell + 1$.

Proof. Since $d = \lambda + \delta - \kappa - \Delta$,

$$d - \ell + 1 = \begin{array}{ll} \delta - \Delta + \frac{\lambda}{\kappa} & \text{if } \lambda \equiv 0 \pmod{\kappa}; \\ \delta - \Delta + 1 + \left\lfloor \frac{\lambda}{\kappa} \right\rfloor & \text{if } \lambda \not\equiv 0 \pmod{\kappa}. \end{array}$$

Our basic assumption is that $\Delta - \delta \geq \frac{\lambda}{\kappa} - \kappa + 1$ or $\kappa \geq \frac{\lambda}{\kappa} + 1 + \delta - \Delta$. Hence if κ divides λ, then

$$\kappa \geq \frac{\lambda}{\kappa} + 1 + \delta - \Delta > \delta - \Delta + \frac{\lambda}{\kappa} = d - \ell + 1,$$

as desired. If κ does not divide λ, then

$$\kappa \geq \frac{\lambda}{\kappa} + 1 + \delta - \Delta > \left\lfloor \frac{\lambda}{\kappa} \right\rfloor + 1 + \delta - \Delta = d - \ell + 1.$$

In either case, $\kappa > d - \ell + 1$ and hence $H_{\kappa,d-\ell+1}$ can be constructed. Since the vertices in $K_{\delta+1}$ have degree δ and $\kappa < \delta$, we can remove the edges of $H_{\kappa,d-\ell+1}$ from $K_{\delta+1}$. ■

Proposition 2.13. *We have $\delta(G) = \delta$ and $\Delta(G) = \Delta$.*

Proof. First,

$$\begin{aligned} \rho_G(u_0) &= \rho_L(u_0) + \rho_M(u_0) \\ &= \ \delta - (d - \ell + 1) \ + (\lambda - \kappa - \ell + 1) \\ &= \delta - d + \lambda - \kappa = \Delta. \end{aligned}$$

Since M has not changed, each vertex of M has degree either δ or $\delta + 1$ and so satisfies our bounds. Note that the vertices of L have the following degrees:

$$\rho_L(u_i) = \begin{cases} \delta & \text{for } \kappa \leq i \leq \delta; \\ \delta - d + l - 1 & \text{for } 0 \leq i < \kappa \text{ and } \kappa \text{ even or } d + l \text{ odd}; \\ \delta - d + l - 2 & \text{for } i = \kappa - 1 \text{ and } \kappa \text{ odd and } d + l \text{ even}. \end{cases} \tag{11}$$

We showed in Case I that adding $e_\ell + 1$ edges to a vertex of degree $\delta - 1$ did not make the degree of the vertex larger than Δ and so no vertex in L has degree larger than Δ. We now must show that we have not decreased the size of degrees of the vertices in U too much. Recall that $\beta_\ell > 0$ and so we add $e_\ell + 1$ edges to $u_{\kappa-1}$. Using Lemma 2.9 and for $0 < i < \kappa$,

$$\begin{aligned}
\rho_G(u_i) &\geq \delta - d + l - 1 + e_\ell \\
&= \delta - d - 1 + l + e_\ell \\
&= \delta - d + \begin{cases} \lambda - \kappa & \text{if } \lambda \not\equiv 1 \pmod{\kappa} \\ \lambda - \kappa - 1 & \text{if } \lambda \equiv 1 \pmod{\kappa} \end{cases} \\
&\geq \delta + \lambda - \kappa - 1 - d \\
&= \delta + \lambda - \kappa - 1 - (\lambda + \delta - \kappa - \Delta) \\
&= \Delta - 1 \geq \delta,
\end{aligned}$$

as desired. ■

Proposition 2.14. *Under the hypotheses in this case, $\lambda(G) = \lambda$.*

Proof. Using Proposition 2.7 with $d = \ell$, there are λ edges joining L to M and hence $\lambda(G) \leq \lambda$. The proof of Proposition 2.14 is the same as the proof that we need to show $\lambda(G) = \lambda$. ■

Proposition 2.15. *Under the hypotheses in this case, $\kappa(G) = \kappa$.*

Proof. Deleting the vertices in U from G leaves $(L - U) \cup M$ and so $\kappa(G) \leq \kappa$. Let S be a set of vertices with $|S| < \kappa$. We will show that $G - S$ is connected.

Since $\kappa(M) = \delta$ and $\kappa < \delta$, we have that deleting S from G leaves M connected. Since $|S| < \kappa$, there is a vertex in U in $G - S$ and so there is a vertex in L adjacent to M in $G - S$. We now show that $L - S$ is connected. To do this we must show that $\delta > 2(\kappa - 1)$.

Suppose $\delta \leq 2(\kappa - 1)$. In this case, recall that $d > \ell$, which using (2) and

(5) is the same as

$$
\begin{aligned}
d &> \ell \\
\lambda+\delta-\kappa-\Delta &> \begin{matrix} \lambda-\kappa+1-\frac{\lambda}{\kappa}, & \text{if } \lambda \equiv 0 \pmod{\kappa}; \\ \lambda-\kappa-\left\lfloor\frac{\lambda}{\kappa}\right\rfloor, & \text{if } \lambda \not\equiv 0 \pmod{\kappa}, \end{matrix} \\
\delta-\Delta &> \begin{matrix} 1-\frac{\lambda}{\kappa}, & \text{if } \lambda \equiv 0 \pmod{\kappa}; \\ -\left\lfloor\frac{\lambda}{\kappa}\right\rfloor, & \text{if } \lambda \not\equiv 0 \pmod{\kappa}, \end{matrix} \\
\Delta-\delta &< \begin{matrix} \frac{\lambda}{\kappa}-1, & \text{if } \lambda \equiv 0 \pmod{\kappa}; \\ \left\lfloor\frac{\lambda}{\kappa}\right\rfloor, & \text{if } \lambda \not\equiv 0 \pmod{\kappa}. \end{matrix}
\end{aligned}
$$

If $\lambda \equiv 0 \pmod{\kappa}$, then

$$1 = 2-1 > \tfrac{2(\kappa-1)}{\kappa} - 1 \geq \tfrac{\delta}{\kappa} - 1 > \tfrac{\lambda}{\kappa} - 1 > \Delta - \delta,$$

an impossibility since $\Delta - \delta \geq 1$. If $\lambda \not\equiv 0 \pmod{\kappa}$, then

$$1 = \left\lfloor \tfrac{2(\kappa-1)}{\kappa} \right\rfloor \geq \left\lfloor \tfrac{\delta}{\kappa} \right\rfloor \geq \left\lfloor \tfrac{\lambda}{\kappa} \right\rfloor > \Delta - \delta,$$

which is the same impossibility as above. Therefore,

$$\delta > 2(\kappa-1) \text{ or } \delta - (\kappa-1) > \kappa - 1.$$

Since $|L-S| \geq (\delta+1)-(\kappa-1) = \delta-(\kappa-1)+1 > (\kappa-1)+1 = \kappa$, at least one vertex, u_i, of $L-U$ is in $L-S$. Also since $|S| \leq \kappa-1$, at least one vertex of U is in $L-S$. Let $u_j \in L-S$ for $j = i$. In L there are the following δ paths from u_i to u_j all of whose interiors are vertex disjoint: $u_i \to u_j$, $u_i \to u_k \to u_j$ where $0 \leq k \leq \delta$ but $k = i$ and $k = j$. If $u_j \in U$, then at most $\kappa-1$ of these paths are not in $L-S$. Therefore, in $L-S$ there are at least $\delta-(\kappa-1) > \kappa-1$ vertex disjoint paths joining u_i and u_j. Hence $L-S$ is connected.

Since $(S\cap M)\cup(S\cap U) \subseteq S$, we have $|S\cap M|+|S\cap U| \leq |S| \leq \kappa-1$ or $|S\cap U| \leq \kappa-1-|S\cap M|$. Also, the number of edges from $L-S$ to M is at least $|U-S| = \kappa-|S\cap U| \geq \kappa - \ \kappa-1-|S\cap M| \ = 1+|S\cap M| \geq 1$. Therefore $L-S$ and $M-S$ are both connected and there is an edge between them. Whence $G-S$ is connected. We can conclude that $\kappa(G) \geq \kappa$. ■

This ends the proof of Theorem 2.1. ■

Main Theorem II: $\Delta - \delta < \frac{\lambda}{\kappa} - \kappa + 1$.

Theorem 2.16. *Given $\kappa < \lambda < \delta < \Delta$ with $\kappa + \Delta < \lambda + \delta$ and*

$$\Delta - \delta < \frac{\lambda}{\kappa} - \kappa + 1, \tag{12}$$

$(\kappa, \lambda, \delta, \Delta)$ is realizable for $n \geq 2\delta + 2$.

Proof. Since $\lambda < \delta$ we know that $n \geq 2\delta + 2$. Thus we only need to construct graphs with these parameters for each $n \geq 2\delta + 2$.

Observation 2.17. *Under our hypotheses, $\lambda > \kappa^2$.*

Proof. If $\lambda \leq \kappa^2$, then using (13), we have $\Delta - \delta < \frac{\lambda}{\kappa} - \kappa + 1 \leq \frac{\kappa^2}{\kappa} - \kappa + 1 = 1$, an impossibility since $\delta < \Delta$. ■

Case I. κ is even.

Let $\lambda = \kappa q + r$ where q and r are integers with $0 \leq r < \kappa$. Let U be the vertices $\{u_0, \ldots, u_{\frac{\kappa}{2}-1}\}$ and let V be the vertices $\{v_0, \ldots, v_{\frac{\kappa}{2}-1}\}$. Let $L = K_{\delta+1} = U \cup \{u_{\frac{\kappa}{2}}, \ldots, u_\delta\}$ and $M = H_{n-\delta-1,\delta} = V \cup \{v_{\frac{\kappa}{2}}, \ldots, v_{n-\delta-2}\}$ where if both n and δ are odd, then $\rho_M(v_{n-\delta-2}) = \delta + 1$. Our graph will be $G = L \cup M$ with λ edges added connecting L and M and with edges deleted in both L and M to maintain the correct maximum degree. Note that G has n vertices where $n \geq 2\delta + 2$.

Lemma 2.18. *Under our assumptions, we have that $\delta > \frac{1}{2}(\lambda + \kappa^2) + 1$ and $\delta > \frac{1}{2}\kappa(q + 2) + 1$.*

Proof. Let $\lambda = \kappa^2 + a$ and $\delta = \lambda + b$ where a and b are positive integers. Hence

$$\begin{aligned}
\delta &= \kappa^2 + a + b \\
&= \frac{1}{2}(\kappa^2 + a + b) + \frac{1}{2}\kappa^2 + \frac{1}{2}(a + b) \\
&> \frac{1}{2}(\kappa^2 + a) + \frac{1}{2}\kappa^2 + 1 = \frac{1}{2}(\lambda + \kappa^2) + 1 \\
&= \frac{1}{2}(\kappa q + r + \kappa^2) + 1 \geq \frac{1}{2}\kappa(q + \kappa) + 1 \geq \frac{1}{2}\kappa(q + 2) + 1.
\end{aligned}$$

The third line gives us the first inequality and the last line the second. ■

We now add λ edges connecting L and M. We do this by adding $q+1$ edges to r vertices in $U \cup V$ and q edges to the other vertices in $U \cup V$. If $r = 0$, then all vertices in $U \cup V$ will have q edges added. If $\kappa \geq 2r$, then $\{u_0, \ldots, u_{r-1}\}$ will have $q+1$ edges added and all the others vertices in $U \cup V$ will have q edges added. If $\kappa < 2r$, then each vertex in U as well as vertices $\{v_0, \ldots, v_{\frac{\kappa}{2}-r-1}\}$ will have $q+1$ edges added. The other vertices in V will have q vertices added. Let q_{u_i} (q_{v_i}) be the number of edges added to vertex u_i (v_i) respectively.

Our goal is to add these edges so that no vertices in U and V will be adjacent, so that $N_M(u_i) \cap N_M(u_j) = \varnothing$ for u_i and u_j in U for $i = j$, and $N_M(u_i) \subseteq N_M(v_i)$ for each $0 \leq i < \frac{\kappa}{2}$, and for all this also to be true with the u's replaced by v's and M replaced by L. We now state this as

Proposition 2.19. *It is possible to join L and M with λ edges such that each edge is adjacent to a vertex in $U \cup V$, no vertices in U and V are adjacent, and no vertex in U is adjacent to $v_{n-\delta-2}$. Furthermore, $N_M(u_i) \cap N_M(u_j) = \varnothing$ for u_i and u_j in U with $i = j$ and $N_M(u_i) \subseteq N_M(v_i)$ for each $0 \leq i < \frac{\kappa}{2}$, and $N_L(v_i) \cap N_L(v_j) = \varnothing$ for v_i and v_j in V with $i = j$ and $N_L(v_i) \subseteq N_L(v_i)$ for each $0 \leq i < \frac{\kappa}{2}$.*

Proof. Choose $u_i \in U$. We need at most $q+1$ vertices in $N_M(v_i)$ to be adjacent to u_i, we need to exclude the other $\frac{\kappa}{2} - 1$ vertices in V, we need to miss any of the other $N_M(u_j)$ where $i = j$, and we wish to avoid $v_{n-\delta-2}$. Thus we must have

$$\begin{aligned}\delta = \left|N_M(v_i)\right| &\geq (q+1) + \left(\frac{\kappa}{2} - 1\right) + \left(\frac{\kappa}{2} - 1\right)(q+1) + 1 \\ &= \left(1 + \frac{\kappa}{2} - 1\right)(q+1) + \left(\frac{\kappa}{2}\right) \\ &= \left(\frac{\kappa}{2}\right)(q+2), \end{aligned} \tag{13}$$

which is true by Lemma 2.18.

We start with u_0. Choose any q (if $\lambda \equiv 0 \pmod{\kappa}$) or any $q+1$ vertices (if $\lambda \not\equiv 0 \pmod{\kappa}$) in $N_M(v_0)$ that are not in $V \cup \{v_{n-\delta-2}\}$. Connect each with u_0. Assume we have chosen edges for u_j, $0 \leq j < i$. Choose q_{u_i} vertices in $N_M(v_i)$ that are not in $V \cup \{v_{n-\delta-2}\}$ or have been previously chosen. (Note we choose q if $i > r$, otherwise choose $q+1$.) This can be done by (13). Connect each of these with u_i.

Follow the same procedure for choosing edges in L for each of the vertices in V. ■

At this point the minimum degree of G is δ and there is a set of edges of size λ that disconnects G and κ vertices whose removal disconnects G. Unfortunately, the maximum degree of G is too large. To see this note that $\kappa \geq 2$ and $r < \kappa$ and so

$$\begin{aligned}
\delta + q_{u_i} &\geq \delta + q \\
&= \delta + \frac{\lambda - r}{\kappa} = \delta + \frac{\lambda}{\kappa} - \frac{r}{\kappa} \\
&> \delta - \frac{r}{\kappa} + (\Delta - \delta + \kappa - 1) \qquad \text{(from 12)} \\
&= \Delta + \kappa - 1 - \frac{r}{\kappa} > \Delta.
\end{aligned}$$

Hence we must remove

$$m_{u_i} = \delta + q_{u_i} - \Delta \tag{14}$$

edges from $u_i \in U$ and we must remove

$$m_{v_i} = \delta + q_{v_i} - \Delta$$

edges from $v_i \in V$. To remove edges from v_i, $0 \leq i < \frac{\kappa}{2}$, choose m_{v_i} vertices, $N(M, i)$, in $N_M(u_i) \subseteq N_M(v_i)$ and remove the edge from each vertex in $N(M, i)$ to v_i. Call this set of edges $E_{M,i}$. Since $\delta - \Delta < 0$, we have that $m_{v_i} < q_{v_i}$ and we know $N_M(u_i) = q_{u_i} \geq q_{v_i} > m_{v_i}$.

Likewise, to remove edges from u_i, $0 \leq i < \frac{\kappa}{2}$, choose m_{u_i} vertices, $N(L, i)$, in $N_L(v_i) \subseteq N_L(u_i)$ and remove the edge from each vertex in $N(L, i)$ to u_i. Call this set of edges $E_{L,i}$. Since $\delta - \Delta < 0$, we have that $m_{u_i} < q_{u_i}$ and so $m_{u_i} \leq q$. Since $N_L(v_i) = q_{v_i} \geq q$, there are enough vertices in $N_L(v_i)$ to choose m_{u_i} vertices.

Proposition 2.20. *The graph $G = L \cup M$ where the edges have been added and removed as above has $\delta(G) = \delta$ and $\Delta(G) = \Delta$.*

Proof. We have just shown that the degree in G of each vertex in $U \cup V$ is Δ. To all other vertices, we have added at most one edge and to some of those we have removed an edge. Hence all vertices of G not in $U \cup V$ have degree either δ or $\delta + 1$ where $\Delta \geq \delta + 1$. ■

It remains to show that G has the correct edge and vertex connectivity. Recall that L and M have edge and vertex connectivity δ. We let $E_L = \bigcup_{0\leq i<\frac{\kappa}{2}} E_{L,i}$, $L' = L - E_L$, and likewise $E_M = \bigcup_{0\leq i<\frac{\kappa}{2}} E_{M,i}$ and $M' = M - E_M$.

Proposition 2.21. *In this case, $\kappa(G) = \kappa$.*

Proof. Let S be a set of $\kappa - 1$ or fewer vertices in G. Recall that L' is L with at most $\frac{1}{2}\kappa q$ edges removed. Hence by Fact 2 and the fact that $\kappa(L) = \delta$, we have that $\kappa(L') \geq \delta - \frac{1}{2}\kappa q$. Since

$$\delta - \tfrac{1}{2}\kappa q \;\; - |S| \geq \delta - \;\; \tfrac{1}{2}\kappa(q+2) \;\; + 1,$$

by Lemma 2.18, $\kappa(L') > |S|$. Likewise, $\kappa(M') > |S|$. Thus both $L' - S$ and $M' - S$ are connected.

Since $|U \cup V| = \kappa$, at least one vertex of $U \cup V$ remains in $G - S$. Since the degree of this vertex in the component it is not in is at least q and since by Observation 2.17, $q \geq \kappa$, there is an edge that connects L' and M' in $G - S$. Therefore $G - S$ is connected. ■

Proposition 2.22. *Under our current hypotheses, $\lambda(G) = \lambda$.*

Proof. To show that $\lambda(G) = \lambda$, assume to the contrary that G has a bond S of fewer than λ edges. Since $|S| < \delta$, we know by Fact 1 that $G - S$ has two components, each with at least $\delta + 1$ vertices.

We must now determine if there is a vertex L that is adjacent to all other vertices in L'. The $\frac{\kappa}{2}$ vertices in U are not such a vertex nor are the vertices in $\bigcup N_{L,i}$ but all other vertices are. By Lemma 2.18,

$$\tfrac{\kappa}{2} + \textstyle\sum_{i=0}^{\frac{\kappa}{2}-1} m_{u_i} \leq \tfrac{\kappa}{2} + \tfrac{\kappa}{2} \cdot q = \tfrac{\kappa}{2}(1+q) < \delta$$

and so there are vertices in L adjacent to all other vertices in L'. Call one such vertex w. Let x be a vertex in $L - U$. So $\rho_L(x) \geq \delta - 1 > |S|$. Since each vertex to which x is adjacent is either w or adjacent to w, there are at least $\delta - 1$ edge-disjoint paths from x to w. Hence, x and w are in the same component of $L' - S$.

Let $u_i \in U$. So $\rho_L(u_i) = \delta - m_{u_i} \geq \delta - q$ and thus there are at least $\delta - q$ edge-disjoint paths from u_i to w. If u_i and w were in different components of $G - S$, then $\delta - q$ edges (one from each path) must be in S. Hence there must be at most $|S| - (\delta - q)$ edges of S in M'. There are at most $\frac{1}{2}\kappa q$ edges of M that are not in M' and so $M' - S$ is missing at most

$$\tfrac{1}{2}\kappa q + |S| - \delta + q$$

edges from M. If this number is less than δ, then $M' - S$ is connected since $\lambda(M) = \delta$. Thus

$$\begin{aligned} \frac{1}{2}\kappa q + |S| - \delta + q \le \left(\frac{1}{2}\kappa + 1\right) q + (\lambda - 1) - \lambda \le \frac{\kappa + 2}{2} \cdot \frac{\lambda - r}{\kappa} \\ \le \frac{\kappa + 2}{2\kappa} \cdot \lambda \le \lambda < \delta, \end{aligned}$$

as desired. Since $M' - S$ is connected and L' has $\delta + 1$ vertices, if $G - S$ has two components, then one must be $L' - S$. Thus u_i and w are in the same component of $G - S$.

At this point we know that $L' - S$ is contained in one of the components of $G - S$;call this component J.

Let E_M be the set of edges deleted from M to get M'. Let $b = \ [L, M] \cap S$ and $c = \ M' \cap S$. Hence $b + c = |S| \le \lambda - 1$.

If $c + |E_M| < \delta$, then $M' - S$ is connected since $\lambda(M) = \delta$. In this case, since $|S| < \lambda$, there is an edge in $[L, M]$ that is not in S and so $L' - S$ is connected and is connected to $M' - S$. Thus $G - S$ is connected. Hence we assume the contrary that

$$c + |E_M| \ge \delta,$$

and will show that this produces a contradiction.

For each vertex $v \in V$, there are at most $\rho_L(v) - 1$ vertices in M to which v is adjacent in M but not in M' and each of these vertices is adjacent to the same vertex in U. Let w be such a vertex in M. Hence w is adjacent to $u_v \in U$ and since $L' - S$ is connected, there is a path in $L' - S$ from u_v to $x \in L$ where x is adjacent to v. Call this path from w to v $P_{w,v}$. Although the edge $\{w, v\}$ is not in M', we can use the path $P_{w,v}$ instead. Hence we will put the edge $\{w, v\}$ back into M'.

Let $|V \cap J| = j$. Note for $v \in V \cap J \subseteq M$, the degree of v in L is either q or $q+1$. Let $U_{V\cap J}$ be the vertices in U associated with those in $V \cap J$. Based on the reasoning in the previous paragraph, let α be the number of edges in $N_M(U_{V\cap J})$ that are not in S. Hence we decrease the size of E_M by α.

We know that before adjusting E_M,

$$|E_M| \le \begin{array}{ll} \frac{1}{2}\kappa(q-1), & \text{if } r \le \frac{1}{2}\kappa; \\ \frac{1}{2}\kappa(q-2) + r, & \text{if } r > \frac{1}{2}\kappa. \end{array} \tag{15}$$

Also, $c + |E_m| \geq \delta$ and $b + c \leq \lambda - 1$ and $\delta \geq \lambda + 1$ and thus

$$b \leq \quad E_M \quad - 2 \tag{16}$$

Using (15) and (16) and the adjusted E_M, we have

$$b \leq \quad \begin{array}{ll} \frac{1}{2}\kappa(q-1) - 2 - \alpha, & \text{if } r \leq \frac{1}{2}\kappa; \\ \frac{1}{2}\kappa(q-2) + r - 2 - \alpha, & \text{if } r > \frac{1}{2}\kappa. \end{array} \tag{17}$$

If $r \leq \frac{\kappa}{2}$, then there are at most $j(q-1)$ edges in $N_M(U_{V \cap J})$ and hence there are at least $j(q-1) - \alpha$ edges in $N_M(U_{V \cap J})$ which are in S. Note also that for each $v \in V$ that is not in J, each of its q edges to L must be in S. Hence

$$b \geq \quad j(q-1) - \alpha \quad + \quad \frac{1}{2}\kappa - j \quad q = \frac{1}{2}\kappa q - j - \alpha.$$

Using this and (17) we have

$$\frac{1}{2}\kappa(q-1) - 2 - \alpha \geq \frac{1}{2}\kappa q - j - \alpha$$

or

$$j \geq \frac{1}{2}\kappa + 2,$$

which is an impossibility since $j \leq |V| \leq \frac{1}{2}\kappa$.

If $r > \frac{\kappa}{2}$, then there are at most $(\kappa - r)(q-1) + (r - \frac{1}{2}\kappa)q$ edges in $N_M(U_V)$. In $V \cap J$, let there be j_1 vertices in the $\kappa - r$ and j_2 vertices in the $r - \frac{1}{2}\kappa$ vertices. Note that $j_1 + j_2 = j$. Now of the edges in $N_M(U_V)$ that are associated with $V \cap J$, there are at most $j_1(q-1) + j_2 q = jq - j_1$ edges, α of which are not in S.

For each vertex in V not in J, there are $\kappa - r - j_1$ which have q edges incident to vertices L and $r - \frac{1}{2}\kappa - j_2$ which have $q+1$ edges incident to vertices in L. All these edges must be in S. Hence,

$$\begin{aligned} b &\geq \quad jq - j_1 - \alpha \quad + (\kappa - r - j_1)q + \quad r - \frac{1}{2}\kappa - j_2 \quad (q+1) \\ &= -j - \alpha + r + \frac{1}{2}\kappa(q-1). \end{aligned}$$

Using this and (17) we have

$$\tfrac{1}{2}\kappa(q-2)+r-2-\alpha \geq -j-\alpha+r+\tfrac{1}{2}\kappa(q-1)$$

or

$$j \geq \tfrac{1}{2}\kappa+2,$$

an impossibility since $j \leq |V| \leq \frac{1}{2}\kappa$. Hence $c+|E_M| < \delta$ and so $G-S$ is connected. ■

We have now shown that if κ is even and $\Delta-\delta < \frac{\lambda}{\kappa}-\kappa+1$, then $(\kappa,\lambda,\delta,\Delta)$ is realizable for $n \geq 2\delta+2$. To show this is true for κ odd, we first construct a realizable graph G for $(\kappa-1,\lambda-1,\delta,\Delta)$. This is possible since $\Delta-\lambda < \frac{\lambda}{\kappa}-\kappa+1$ implies that $\Delta-\lambda < \frac{\lambda-1}{\kappa-1}-(\kappa-1)+1$.

It is not difficult to show that there is a vertex in L and another in M for which no edges have been added or removed in the construction for G. Adding an edge joining these two vertices will give a graph $G^{'}$ with $\kappa(G^{'})=\kappa$ and $\lambda(G^{'})=\lambda$ and which does not change the minimum or maximum degree. The proof of this is straightforward and omitted.

3 $\kappa = 2$

The remaining sections will be devoted to the description of results we have found for other choices of $(\kappa,\lambda,\delta,\Delta)$. We do not offer proofs for all of these claims, but for those without complete proofs we do give explanations.

We next state our results for the case when $\kappa+\Delta < \lambda+\delta$ and $\kappa=2$. For several of the remaining cases we have found that $\kappa=2$ is a special subcase and so we gather all the remaining $\kappa=2$ cases here. For all restrictions on $(\kappa,\lambda,\delta,\Delta)$ beyond $\kappa+\Delta < \lambda+\delta$ except for the restriction $\kappa < \lambda < \delta < \Delta$, the constructions of the graphs that realize $(2,\lambda,\delta,\Delta)$ differ from those that realize $(\kappa,\lambda,\delta,\Delta)$, $\kappa > 2$. The subcases for $\kappa=2$ that we consider are the same as those for $\kappa > 2$. We first state a fact from Boesch [1].

Fact 3.1. *For any noncompete graph with more than one vertex, $n \geq 2\delta+2-\kappa$.*

Theorem 3.2. *If $\delta > 2$ is odd, then the sequence $(2,\delta,\delta,\delta)$ is not realizable. If δ is even, then $(2,\delta,\delta,\delta)$ is realizable for $n \geq 2\delta+2$.*

Proof. Let $G=$ be the graph $L \cup \{u,v\} \cup M$ where G realizes $(2,\delta,\delta,\delta)$ and $\{u,v\}$ is a cutset of size two. We have the following inequalities:

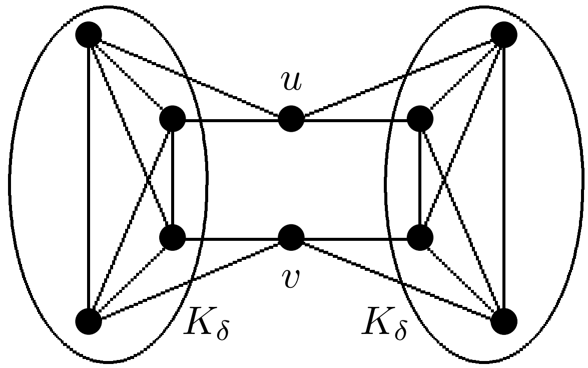

Figure 2: Graph realizing $(2, \delta, \delta, \delta)$ for $n = 2\delta + 2$ with $\delta = 4$

$$\begin{gathered}
\rho_L(u) + \rho_L(v) \geq \delta, \\
\rho_M(u) + \rho_M(v) \geq \delta, \\
\delta \geq \rho_L(u) + \rho_M(u) \geq \delta - 1, \\
\delta \geq \rho_L(v) + \rho_M(v) \geq \delta - 1.
\end{gathered}$$

Thus $2\delta \geq \rho_L(u) + \rho_M(u) + \rho_L(v) + \rho_M(v) \geq 2\delta$. Note that this means that $\delta = \rho_G(u) = \rho_L(u) + \rho_M(u)$ and so there is no edge between u and v. Let $a = \min\{\rho_L(u), \rho_M(u)\}$ and $b = \min\{\rho_L(v), \rho_M(v)\}$. Note that $a \leq \frac{\delta}{2}$ and $b \leq \frac{\delta}{2}$. If either a or b is strictly less than $\frac{\delta}{2}$, then $a + b < \delta$ and so either $\rho_G(u) < \delta$, or $\rho_G(v) < \delta$, or we can remove $\delta - 1$ edges to disconnect G. All these are contradictions and so a and b are both $\frac{\delta}{2}$ and so $\frac{\delta}{2} = \rho_L(u) = \rho_M u = \rho_L(v) = \rho_M(v)$.

We have now shown that δ must be even. We next show that $n_G \geq 2\delta + 2$. If a vertex in L, say w, is not adjacent to both u and v, then $\rho_L(w) = \delta - 1$ and so $n_L \geq \delta$ and therefore, $n_G = n_L + n_M + 2 \geq 2\delta + 2$ since $n_M \geq n_L$. If each vertex in L is adjacent to both u and v, then since $\rho_L(u) = \frac{\delta}{2}$, we have that $n_L = \frac{\delta}{2}$. If $w \in L$, then $\delta = \rho_G(w) = \rho_L(w) + 2 \leq (\frac{\delta}{2} - 1) + 2$ and so $\delta \leq 2$, a contradiction. Thus we have shown that $n_G \geq 2\delta + 2$.

All that remains is for us to show that if δ is even, then $(2, \delta, \delta, \delta)$ is realizable for $n \geq 2\delta + 2$. There are two cases: $n = 2\delta + 2$ and $n > 2\delta + 2$.

For the first case, let $L = K_\delta$ with vertices $\{u_0, \ldots, u_{\delta-1}\}$, let N_2 have vertices $\{u, v\}$, let $M = K_\delta$ with vertices $\{v_0, \ldots, v_{\delta-1}\}$, and let G be

$$L \cup N_2 \cup M$$

where u is adjacent to u_i and v_i for $0 \leq i < \frac{\delta}{2}$ and v is adjacent to u_i and v_i for $\frac{\delta}{2} \leq i < \delta$. See Figure 2 for an example where $\delta = 4$. It is clear that $\kappa(G) = 2$, $n(G) = 2\delta + 2$, and $\delta(G) = \Delta(G) = \delta$. We only need show that $\lambda(G) = \delta$.

Let S be a set of $\delta - 1$ edges in G. Note that since L and M are both K_δ, if fewer than $\delta - 1$ edges are removed from either, the resulting graph is still connected. Since the size of $[u, L]$, $[v, L]$, $[u, M]$, and $[v, M]$ is $\frac{\delta}{2}$, in $G - S$ one of u or v is connected to both L and M and the other is connected to at least one of L or M. Thus if $|S \cap L| < \delta - 1$ and $|S \cap M| < \delta - 1$, then $G - S$ is connected.

Suppose next that all edges of S are in L or all are in M; say that all are in L. Let w and x be vertices in G. Clearly neither w nor x is in L, then there is a path between them in $G - S$. If w is in L but x is not, then w is adjacent to either u or v in $G - S$ and hence there is a path from w to x in $G - S$. If both w and x are in L, then each is adjacent to one of u or v. Since there is path from u to v in $G - S$, there must be a path from w to x in $G - S$. Therefore, $\lambda(G) = \delta$.

For the second case, let L and N_2 be as in the first case and $M = H_{n-\delta-2,\delta}$ with vertices $\{v_0, \ldots, v_{n-\delta-3}\}$. We let $G = L \cup N_2 \cup M$ where u and v are adjacent to the same vertices in L and M as in the first case. Note that since $n > 2\delta + 2$, we have that $n - \delta - 2 > \delta$ and so there are enough vertices in M for this construction. To make G regular, we remove the following edges in M: $\{e_{2i,2i+1} : 0 \leq i < \frac{\delta}{2}\}$. For an example of such a graph, see Figure 3 where the Harary graph has the edges $e_{0,1}$ and $e_{2,3}$ removed and where $\delta = 4$ and $n = 14$.

Once again, it is clear that $\kappa(G) = 2$ and $\delta(G) = \Delta(G) = \delta$. We only need show that $\lambda(G) = \delta$. As in the first case, we let S be a set of $\delta - 1$ edges of G and note that if all the edges of S are in L, then $G - S$ is connected. Also, we have that one of u or v is adjacent to both L and M in $G - S$ and the other is adjacent to at least one of L or M in $G - S$. Finally, if $M - S$ is connected, then $G - S$ is connected.

Recall that by assumption δ is even and so $\frac{\delta}{2}$ exists and is either even or odd. Note that since $\lambda(H_{n-\delta-2,\delta}) = \delta$ and we have removed $\frac{\delta}{2}$ edges from the Harary graph to form M, the number of edges of S in M, denoted s_M,

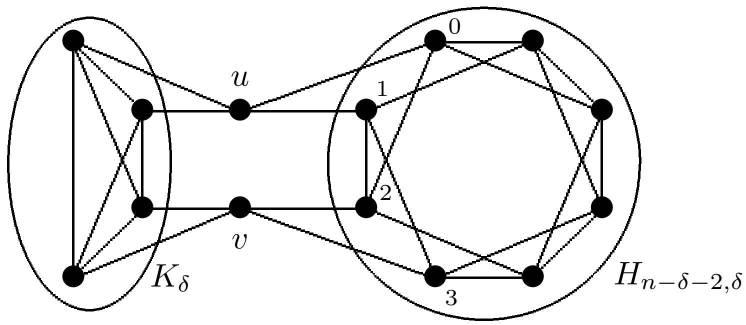

Figure 3: Graph realizing $(2,\delta,\delta,\delta)$ for $n > 2\delta+2$ with $\delta = 4$

must satisfy $\frac{\delta}{2} \leq s_M < \delta$. But also note that if the edges from N_2 incident to v_{2i} and v_{2i+1} are not both in S and both are incident to the same vertex in N_2, then the edge $e_{2i,2i+1}$ that was deleted from the Harary graph may be restored to M since there is a path in $G-S$ from v_{2i} to v_{2i+1}. Let a be the number of pairs of edges that are not in S of this type. Thus we now have that

$$\frac{\delta}{2} + a \leq s_M. \tag{18}$$

If $\frac{\delta}{2}$ is even, then there are $\frac{\delta}{2} - a$ pairs that are not of this type and if $\frac{\delta}{2}$ is odd, there are $\frac{\delta}{2} - a + 1$ that are not of this type. For $\frac{\delta}{2}$ even and for each pair not of this type, there must be at least one edge in S for each pair. Hence $s_M + (\frac{\delta}{2} - a) < \delta$ or $s_M < \frac{\delta}{2} + a$ which contradicts (18).

If $\frac{\delta}{2}$ is odd, then since the end-vertices of the deleted edge $e_{\frac{\delta}{2}-1,\frac{\delta}{2}}$ are adjacent to u and v, we know that only $\frac{\delta}{2} - 1 - a$ edges must be in S and so $s_M < \frac{\delta}{2} + 1 + a$ and hence by (18) we have $s_m = \frac{\delta}{2} + a$. But this accounts for all the edges of S: either the edge is in M or the edge is incident to u or v and a vertex in M. Hence, there is a path in $G-S$ through L that connects u and v. Thus either both edges $e_{u,\frac{\delta}{2}-1}$ and $e_{v,\frac{\delta}{2}}$ are not in S and so a is one greater or at least one of the edges is in S. In either case, we are back to the $\frac{\delta}{2}$ even case and we have that $G-S$ must be connected. Therefore $\lambda(G) = \delta$ and we have shown that $(2,\delta,\delta,\delta)$ for δ even is realizable for all $n \geq 2\delta+2$. ■

Theorem 3.3. *If $\kappa = 2$, $\kappa + \Delta < \lambda + \delta$, and $\lambda = \delta$, then $(2,\delta,\delta,\Delta)$ is not* *realizable if* $n < 2\delta + 1$. *If, in addition,* $\Delta - \delta < \frac{1}{2}\delta - 1$, *then* $(\kappa, \delta, \delta, \Delta)$ *is not realizable if* $n < 2\delta + 2$.

Proof. Suppose G realizes $(2, \delta, \delta, \delta)$ where δ and Δ meet the hypothesis of this theorem. Let $K = \{u, v\}$ be the cut set of G where G realizes $(2, \delta, \delta, \Delta)$. Let $G - K = L \cup M$ where $|L| < |M|$. Let $w \in L$. Since $\delta \leq \rho_G(w) = \rho_L(w) + 2$, we have $\rho_L(W) \geq \delta - 2$ and so $n_L \geq \delta - 1$ and thus $n_M \geq \delta - 1$. Hence $n_G \geq 2\delta$. If $n_G = 2\delta$, then $L = M = K_{\delta-1}$ and so every vertex of L and M is adjacent to both u and v. Thus $\Delta \geq \rho_G(u) \geq \rho_L(u) + \rho_M(u) = (\delta - 1) + (\delta - 1) = 2\delta - 2$. But since $\kappa + \Delta < \lambda + \delta$ or $2 + \Delta < 2\delta$, we have the contradiction that $\Delta < 2\delta - 2$. Therefore $n_G \geq 2\delta - 1$.

Next, suppose $\Delta - \delta < \frac{\delta}{2} - 1$ and G realizes $(2, \delta, \delta, \delta)$. If $n_G = 2\delta + 1$, then $L = K_{\delta-1}$ and $n_M = \delta$. Note that u and v are adjacent to each vertex of L. Thus

$$\begin{aligned} 2\Delta \geq \rho_G(u) + \rho_G(v) &\geq (\delta - 1 + \rho_M(u)) + (\delta - 1 + \rho_M(v)) \\ &= 2\delta - 2 + (\rho_M(u) + \rho_M(v)). \end{aligned}$$

Let $x \in M$. Hence

$$\begin{aligned} \delta \leq \rho_G(x) &= \rho_M(x) + \rho_K(x) \\ &\leq \delta - 1 + \rho_K(x). \end{aligned}$$

Thus $1 \leq \rho_K(x)$ for all $x \in M$. Hence $\rho_M(u) + \rho_M(v) \geq n_M = \delta$. We have $2\Delta \geq 2\delta - 2 + \delta$ or $\Delta - \delta \geq \frac{\delta}{2} - 1$, a contradiction. ■

Now we will examine the case when $\kappa = 2$ and $\lambda < \delta = \Delta$ and δ is even. By Fact 1.13, $n \geq 2\delta + 2$.

We argue that given $\kappa = 2$ and $\lambda < \delta = \Delta$ all even, $(2, \lambda, \delta, \delta)$ is realizable for $n \geq 2\delta + 2$. Let $\kappa = 2$ and let $\lambda < \delta = \Delta$ be even. Consider the case when $n = 2\delta + 2$. Let $L = K_\delta$ with vertices $\{u_0, u_1, \ldots, u_{\delta-1}\}$, let $K = H_{2,0}$ with vertices $\{x_0, x_1\}$, and let $M = K_\delta$ with vertices $\{v_0, v_1, \ldots, v_{\delta-1}\}$. The graph of interest G is $L \cup K \cup M$.

Observe that $G - K$ disconnects L and M. Thus $\kappa(G) \leq 2$. Removing any other two vertices from L, M, or L and M does not disconnect G and hence $\kappa(G) = 2$.

Since $L = K_\delta$, $\rho_L(u_i) = \delta - 1$ for $u_i \in L$. Let $\rho_L(x_0) = \frac{\lambda}{2}$ and let $\rho_L(x_1) = \delta - \frac{\lambda}{2}$. Then there are $\rho_L(x_0) + \rho_L(x_1) = \frac{\lambda}{2} + (\delta - \frac{\lambda}{2}) = \delta$ edges from K to L. Arrange the δ edges so that each of $u_0, \ldots, u_{\delta-1}$ is adjacent to either x_0 or x_1. Then by construction $\rho(u_i) = (\delta - 1) + 1 = \delta$ for all u_i. We arrange the edges from K to M in a similar fashion. Since $M = K_\delta$, $\rho_M(v_i) = \delta - 1$. Let $\rho_M(x_0) = \delta - \frac{\lambda}{2}$ and $\rho_M(x_1) = \frac{\lambda}{2}$. Then there are $\rho_M(x_0) + \rho_M(x_1) = (\delta - \frac{\lambda}{2}) + \frac{\lambda}{2} = \delta$ edges from K to M. Arrange the δ edges so that each of $v_0, \ldots, v_{\delta-1}$ is adjacent to either x_0 or x_1. Then $\rho(v_i) = (\delta - 1) + 1 = \delta$. Also, $\rho(x_0) = \rho_L(x_0) + \rho_M(x_0) = \frac{\lambda}{2} + (\delta - \frac{\lambda}{2}) = \delta$ and $\rho(x_1) = \rho_L(x_1) + \rho_M(x_1) = (\delta - \frac{\lambda}{2}) + \frac{\lambda}{2} = \delta$. Thus $\delta(G) = \delta$. Furthermore, $\Delta(G) = \delta(G) = \delta$ by construction.

We want to show that $\lambda(G) = \lambda < \delta$. Since all vertices of G have degree δ and $\delta > \lambda$ by assumption, removing all of the edges incident to any one vertex disconnects G but removes greater than λ edges. Consider $\rho_L(x_0) = \frac{\lambda}{2}$ and $\rho_m(x_1) = \frac{\lambda}{2}$. then $\rho_L(x_0) + \rho_M(x_1) = \lambda$ and since removing the $\rho_L(x_0)$ edges from K to L disconnects L from x_0, removing the $\rho_M(x_1)$ edges from K to M disconnects M from x_1, and x_0 and x_1 are not connected in $K = H_{2,0}$, removing $\rho_L(x_0)$ and $\rho_M(x_1)$ disconnects G. A similar argument holds for $\rho_L(x_1) = \frac{\lambda}{2}$ and $\rho_M(x_0) = \frac{\lambda}{2}$.

We can conclude that G is realizable for $n = 2\delta + 2$.

Now consider the case when $n > 2\delta + 2$. Let $L = K_\delta$ with vertices $\{u_0, u_1, \ldots, u_{\delta-1}\}$, $K = H_{2,0}$ with vertices $\{x_0, x_1\}$, and $M = H_{n-\delta-2,\delta}$ with vertices $\{v_0, v_1 \ldots, v_{n-\delta-3}\}$. The graph of interest$G$ is $L \cup K \cup M$.

As in the case when $n = 2\delta + 2$, $\kappa(G) = \kappa$. Let $\rho_L(x_0) = \frac{\lambda}{2}$ and let $\rho_L(x_1) = \delta - \frac{\lambda}{2}$. Arrange the δ edges from L to K so that each of the vertices of G in L is adjacent to x_0 or x_1. Then $\rho(u_i) = (\delta - 1) + 1 = \delta$. Let $\rho_M(x_0) = \delta - \frac{\lambda}{2}$ and let $\rho_M(x_1) = \frac{\lambda}{2}$. Notice that $\rho_L(x_0) + \rho_M(x_0) = \delta$ and $\rho_L(x_1) + \rho_M(x_1) = \delta$. Arrange each of the δ edges from K to M so that any of $v_0, \ldots, v_{n-\delta-3}$ has at most one edge connecting it to K. Note since $n - \delta - 2 > \delta$, not all of v_i will be adjacent to x_0 or x_1. Let S be the set of δ-many vertices of M that are connected to x_0 or x_1. For every $v_i \in S$, arrange the edges from K to M so that v_i is adjacent to v_{i+1}. Then each vertex in S has degree $\delta + 1$. Remove the edge between v_i and v_{i+1} in M for all $v_i \in S$ except $v_{\delta-1}$. Since δ is even, each vertex in S now has degree δ. Thus $\delta(G) = \delta$. Furthermore $\Delta(G) = \delta$.

As in the case when $n = 2\delta + 2$, $\lambda(G) = \delta$.

Thus G is realizable for $n \geq 2\delta + 2$. ■

4 $\kappa > 2$

We will first consider the case when $\lambda = \delta$ and $\kappa > 2$. For $(\kappa, \delta, \delta, \Delta)$ we have that $n \geq 2\delta - \kappa + 2$ from previous results.

We claim that for $(2, \delta, \delta, \Delta)$ and $\Delta - \delta \leq \lceil \frac{\delta}{2} \rceil - 2$, $n \geq 2\delta + 2$. Furthermore, for $(\kappa, \delta, \delta, \Delta)$, $n \geq 2\delta - \kappa + 2 + c$ where $0 \leq c \leq \kappa$. When $c = 0$, $n \geq 2\delta - \kappa + 2$ and the number of choices for Δ for which $n \geq 2\delta - \kappa + 2$ is realized is $\kappa - 2$. Furthermore, the Δ's that are realizable for $n \geq 2\delta - \kappa + 2$ are the largest choices for Δ. For $n \geq 2\delta - \kappa + 2$, the graph is $L = K_{\frac{2\delta-\kappa+2-\kappa}{2}}$, $K = H_{k,a}$, and $M = K_{\frac{2\delta-\kappa+2-\kappa}{2}}$ where $0 \leq a \leq \kappa - 2$. Here there are m edges from M to K and ℓ edges from L to K where

$$m = \ell = \kappa \quad \frac{2\delta - \kappa + 2 - \kappa}{2}$$

with m and ℓ divided as evenly as possible amongst the κ cut vertices.

For $0 \leq c \leq \kappa$, the number of Δ's that are realizable for each c is the number of Δ's such that $\lfloor \frac{m}{\kappa} \rfloor + \lfloor \frac{\ell}{\kappa} \rfloor \leq \rho(x_i) \leq \lfloor \frac{m}{\kappa} \rfloor + (\Delta - \lfloor \frac{m}{\kappa} \rfloor)$ where we now use $L = K_{\lfloor \frac{2\delta-\kappa+2-\kappa+c}{2} \rfloor + f}$, $K = H_{\kappa,0}$, and $M = K_{\lceil \frac{2\delta-\kappa+2-\kappa+c}{2} \rceil + g}$. Let ℓ be the number of edges from L to K and let m be the number of edges from M to K. Then

$$m = \quad \delta - \left(\left\lfloor \frac{2\delta - \kappa + 2 - \kappa + c}{2} \right\rfloor - 1 + g \right) \quad \cdot \quad \frac{2\delta - \kappa + 2 - \kappa + c}{2}$$

and

$$\ell = \quad \delta - \left(\left\lfloor \frac{2\delta - \kappa + 2 - \kappa + c}{2} \right\rfloor - 1 + f \right) \quad \cdot \left(\left\lfloor \frac{2\delta - \kappa + 2 - \kappa + c}{2} \right\rfloor \right)$$

and $0 \leq g - f \leq \lfloor \frac{\kappa}{2} \rfloor$.

When $c > 0$, we increase g as much as possible before increasing f to define L and M. Note $0 \leq c \leq \kappa$ because once $n = 2\delta + 2$, G is realizable.

For any choice of Δ and for every choice of c with $0 \leq c \leq \kappa$, there will be a specific number of sizes n for G that are realizable for $2\delta - \kappa + c$. Given κ, let m be the $(\kappa^2 - 3\kappa + 1)^{\text{th}}$ largest choice of δ. Then for all choices of δ larger than m such that $\delta \equiv m \pmod{\kappa}$, the number of choices of Δ such that the minimum size of the graph is $2\delta - \kappa + 2 + c$ for each $c > 0$ is one more than the previous choice of δ such that $\delta \equiv m \pmod{\kappa}$. In this way, the distribution of minimal sizes for a graph to be realizable given any choices of Δ repeats mod κ. For any parameters, the largest minimum size of a graph that realizes the parameters is $2\delta + 2$ since $0 \leq c \leq \kappa$.

We will now present what we believe to be a correct method for the construction of the graph realizing $(\kappa, \delta, \delta, \delta)$ where $\kappa \geq 2$ and both κ and δ even. For $n = 2\delta + 2$, let $L = K_{\frac{n-\kappa}{2}}$, $K = H_{\kappa,c}$, and $M = K_{\frac{n-\kappa}{2}}$ where $c = \delta - (\frac{\alpha+\beta}{\kappa})$ where α is the number of edges from K to L and β is the number of edges from K to M. Note in this case that $\alpha = (\delta - (\frac{1}{2}(n-\kappa) - 1)(\frac{1}{2}(n-\kappa)) = \beta$. Our graph of interest is $G = L \cup_\alpha K \cup_\beta M$ where $L \cup_\alpha K$ is the union of L and K with α edges and $K \cup_\beta M$ is the union of K and M with β edges.

For $n \geq 2\delta+3$ we will have $G = \cup_\alpha K \cup_\beta M$ where $L = H_{a+1,a}$, $K = H^d_{\kappa,c}$, and $M = H_{n-\kappa-a,b}$. Here α and β are defined as above. For $n = 2\delta+3$, let $a = \frac{1}{2}(n-\kappa-1) - 1$ so that $a+1 = \frac{1}{2}(n-\kappa-1)$. Note L in this case is the same as in the case where $n = 2\delta+2$. Let $b = a$ and let $\beta = (\delta-a)(n-\kappa-a)$. Let $c = \delta - \lfloor \frac{1}{\kappa}(\alpha-\beta) \rfloor$. Let d be the number of vertices of K that have degree $\delta - \lfloor \frac{1}{\kappa}(\alpha-\beta) \rfloor + c$.

We now have two cases. Case I: Suppose κ divides α. Keep L as in the case when $n = 2\delta+3$ and increase M as n increases. Let α, β, c, d, and b be defined as above such that only β, c, and d change as n increases. Case II: Suppose $\alpha = m \cdot \kappa + l$ for some integers m and l. For $n = 2\delta+4$, let $L = H_{n-\kappa-(2\delta+3-\kappa-a),a}$ and $M = H_{(2\delta+3)-\kappa-a,b}$. For $2\delta+5$, increase M by one vertex. For $2\delta+6$, increase L by one vertex. Continue in this manner until $\alpha = (m+1)\kappa$. Now revert to Case I. Let α, β, c, d, and b be defined as above.

Continue in this way until $c = 0$ and $d = 0$. Note in this case $\frac{\alpha+\beta}{\kappa} = \delta$. Redefine L and and M and repeat the algorithm.

5 Conjectures and Future research

Future research following these results is needed to conclude both the $\kappa = 2$ and $\kappa > 2$ cases. For $\kappa = 2$ with $\lambda = \delta < \Delta$, we know that the maximum of all minimums is $2\delta + 2$. The difficulty is that as c changes, the minimum changes and although we can predict how any choices of Δ will correspond with each c, we do not know why this happens. A conclusive analysis will include proof as to why the distribution of Δ's for each c repeats mod κ and how we can predetermine the distribution of choices of Δ for each c before the $(\kappa^2 - 3\kappa + 1)^{\text{th}}$ choice of δ given κ.

For $\kappa = 2$ we have the following conjectures:

Conjecture 1. *Given* $2 < \lambda < \delta = \Delta$ *with* $\kappa + \Delta < \lambda + \delta$ *with* λ *odd and* δ *odd,* $(2, \lambda, \delta, \delta)$ *is realizable for* $n \geq 2\delta + 2$*.*

Conjecture 2. *Given* $2 < \lambda < \delta = \Delta$ *with* $\kappa + \Delta < \lambda + \delta$ *with* λ *even and* δ *odd,* $(2, \lambda, \delta, \delta)$ *is realizable for* $n \geq 2\delta + 2$*.*

Conjecture 3. *Given* $2 < \lambda < \delta = \Delta$ *with* $\kappa + \Delta < \lambda + \delta$ *with* λ *odd and* δ *even,* $(2, \lambda, \delta, \delta)$ *is not realizable.*

Other cases to be completed for $\kappa > 2$ are the regular cases both when $\lambda = \delta$ and when $\lambda < \delta$. Other research will include the non-regular case where $\lambda = \delta$. Our research has completed the strict inequality case and has made conjectures for the cases when $\kappa = 2$, $\delta = \Delta$, and $\lambda = \delta$.

VI. References.

[1] F. T. Boesch, Lower bounds on the vulnerability of a graph, *Networks* **2** (1972), 329-340.

[2] F. T. Boesch and C. L. Suffel, Realizability of p-point graphs with prescribed minimum degree, maximum degree, and line connectivity, *J. Graph Theory* **4** (1980), *no 4*, 363-370.

[3] F. T. Boesch and C. L. Suffel, Realizability of p-point graphs with prescribed minimum degree, maximum degree, and point connectivity, *Discrete Applied Mathematics* **3** (1981), 9-18.

[4] F. T. Boesch and C. L. Suffel, Realizability of p-point, q-line graphs with prescribed point connectivity, line connectivity, or minimum degree, *Networks* **12** (1982), 341-350.

[5] G. Chartrand and F. Harary, Graphs with prescribed connectivities, *Theory of Graphs*, Proc. Tihany 1966, (ed. P. Erdős and G. Katona) Acad. Press (1968), 64-67.

[6] D. DiMarco, Realizability of p-point, q-line graphs with prescribed maximum degree and line connectivity or minimum degree, *Networks* **36** (2000), *no. 1*, 64-67.

[7] D. DiMarco, Realizability of p-point, q-line graphs with prescribed maximum degree, and point connectivity, *Ars Combinatoria* **61** (2001), 137-147.

[8] D. DiMarco, Realizability of p-point, q-line graphs with prescribed maximum degree and line connectivity, *Ars Combinatoria* **65** (2002), 121-128.

[9] D. DiMarco, Realizability of p-vertex, q-edge graphs with prescribed vertex connectivity and minimum degree, *Journal of Combinatorial Mathematics and Combinatorial Computing.* **40** (2002), 5-15.

[10] D. Dimarco, Realizability of connected, separable, p-point, q-line graphs with prescribed minimum degree and line connectivity, *Ars Combinatoria* **89** (2008), 299-308.

[11] F. Harary, The maximum connectivity of a graph, *Proc. Nat. Acad. Sci. U. S. S.* **48** (1962), 202-210.

[12] L. W. Kazmierczak, F. Boesch, D. Gross, and C. Suffel, Realizability results involving two connectivity parameters, *Ars Combinatoria* **82** (2007), 181-191.

[13] D. B. West, *Introduction to Graph Theory*, Prentice Hall, 1996.

[14] H. Whitney, Congruent graphs and the connectivity of graphs, *American Journal of Mathematics* **54** (1932), 150-168.