Runs in Permutations

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1 Introduction

Let $\sigma \in S_n$ be a permutation of length n. A run of length ℓ with a distance of d and a rise of r is a sequence of ℓ numbers in σ, all at a fixed distance d with the difference between successive numbers all being r. A run of length ℓ is called an ℓ-run. For example, 124635 contains 246 as 3-run of rise 2 and distance 1, while 162435 contains 123 as a 3-run of rise 1 and distance 2.

Let $S^{\ell}_{r,d}(n, k)$ denote the number of permutations in S_n which contain exactly k runs of length ℓ with a rise of r and a distance of d. Our goal is to derive formulas which evaluate $S^{\ell}_{r,d}(n, k)$ for all values of ℓ, r, n, k when $d = 1$.

Runs in permutations and similar topics have been studied by various authors under different names. Hegarty [4] examined permutations of finite abelian groups which avoid what he called progressions. Riordan [5] studied 3-runs (which he called 3-sequences) and derived a formula to compute the number of permutations containing x 3-runs with rise of 1 and a distance of 1. Dymacek [3] investigated 3-runs with $d = 1$ and $r = 1$ or $r = 2$. To our knowledge there has not been a formula produced for the cases where both $r > 1$ and $d > 1$.

2 Definitions

Let $[n] = \{1, \dots, n\}$.

Definition 1. A sequential partition of $[n]$ is a set of disjoint sequences of integers whose union (taking the sequences as sets) is $[n]$.

Sequential partitions will be written as $[s_1, \dots, s_k]$ where each s_i is a sequence whose elements are disjoint from the others. For example $[(1, 2, 3), (4, 5)]$ and $[(1, 2, 4), (5, 3)]$ are two sequential partitions of $\{1, 2, 3, 4, 5\}$. The sequences in a sequential partition are called parts. Let $\mathcal{S}_n$ denote the set of all sequential partitions of $[n]$. Parts containing only one part are called trivial parts. For convenience in writing sequential partitions of $[n]$, we will omit the trivial parts as they can be inferred from the rest. That is, $[(1, 2), (4, 5, 3), (6)]$ will be written $[(1, 2), (4, 5, 3)]$ with the (6) implied. Let $\mathcal{A}_n = \{[(i, j)] \mid 1 \le i, j \le n, i \ne j\}$, the set of all sequential partitions with exactly one nontrivial part which has length 2. The elements of $\mathcal{A}$ are called atomic partitions. Notice that if a sequential partition in $\mathcal{S}_n$ contains only one part, that part must necessarily be a permutation of $[n]$. Furthermore, any permutation, $\sigma \in S_n$ can be used to form a sequential partition σ as its only part. Thus, there is a one-to-one correspondence between permutations and sequential partitions with one part. Because of this when we refer to a permutation $\sigma \in S_n$ we mean the corresponding sequential partition in $\mathcal{S}_n$. Finally, for a given n, let $\mathbf{0} = [(1), \dots, (n)]$.

For any two sequences s and s^*, we say that s is contained in s^* if the sequence s appears in the sequence s^*. So, $(1, 2)$ is contained in $(3, 1, 2, 4)$ but not $(1, 3, 2, 4)$. A sequence s is a called a subpart of a sequential partition π if s is contained in a part of π. For example, $(2, 3)$ is not a part of $[(1, 2, 3), (4, 5)]$ but it is a subpart of $[(1, 2, 3), (4, 5)]$. We say a sequential partition π_1 is *below* another sequential partition π_2, denoted $\pi_1 \le \pi_2$, if each part of π_1 is a subpart of π_2. If π_1 is below π_2, we also say that π_2 is *above* π_1. Note that since parts of π partition $[n]$ it is necessarily true that all the trivial parts of π_1 are contained in some part of

π_2. Thus, to prove $\pi_1 \leq \pi_2$ it is sufficient to show that all nontrivial parts of π_1 are subparts of π_2. We use $\pi_1 < \pi_2$ to mean $\pi_1 \leq \pi_2$ and $\pi_1 \neq \pi_2$. If $\pi_1 \leq \pi_2$, we define the interval $[\pi_1, \pi_2] = \{\pi \in \mathcal{S}_n : \pi_1 \leq \pi \leq \pi_2\}$. Let $D(\pi)$ denote the down-set of π, the set of all sequential partitions below π and let $U(\pi)$ denote the up-set of π, the set of all sequential partitions above π. We will see that $\mathcal{S}_n$ under the $\leq$ relation forms a partially ordered set.

Since $\mathbf{0}$ is composed of only trivial parts, we have $\mathbf{0} \leq \pi$ for every $\pi \in \mathcal{S}_n$. It is easy to see that for any $\beta \in \mathcal{A}$ there exists no $\pi \in \mathcal{S}_n$ such that $\mathbf{0} < \pi < \beta$. Hence, atomic partitions are atoms $\mathcal{S}_n$. It is also clear that for any $\sigma \in S_n$ there exists no $\pi \in \mathcal{S}_n$ such that $\sigma < \pi$. Thus, permutations are are the maximal elements of $\mathcal{S}_n$.

Definition 2. Two sequential partitions π_1 and π_2 are called compatible if there exists a sequential partition π above both π_1 and π_2.

For example, $[(1,2,3)]$ and $[(2,3,4)]$ are compatible, as $[(1,2,3,4)]$ is above them both . The sequences $[(2,3)]$ and $[(3,2)]$ are not compatible. To see this, suppose they were both below some sequential partition π. Then π must have both (2,3) and (3,2) as subparts. But, this is impossible as that would necessarily imply that 3 or 2 appears in two places in π. Notice that any two sequential partitions in an interval $[\mathbf{0}, \pi]$ are compatible.

Definition 3. For any $\pi \in \mathcal{S}_n$ define the support of π with $\text{supp}(\pi) = \{\beta \in \mathcal{A}_n : \beta \leq \pi\}$, that is the set of atomic partitions below π.

We now show that for any $\pi \in \mathcal{S}_n$, the poset induced on $[\mathbf{0}, \pi]$ by $\leq$ is isomorphic to the lattice of subsets of $\text{supp}(\pi)$ ordered via subset inclusion.

Lemma 2.1. For all sequential partitions π_1 and π_2, we have $\pi_1 \leq \pi_2$ if and only if $\text{supp}(\pi_1) \subseteq \text{supp}(\pi_2)$.

Proof. Suppose $\pi_1 \leq \pi_2$. Then any part of π_1 is a subpart of π_2. Suppose $\beta = [(i,j)] \in \text{supp}(\pi_1)$. Then (i,j) is a subpart of π_1, and therefore (i,j) is contained in a part of π_1 which is a subpart of π_2. Thus, (i,j) is a subpart of π_2. Since (i,j) is the only nontrivial part of β, this implies $\beta \leq \pi_2$, so $\beta \in \text{supp}(\pi_2)$.

Suppose $\text{supp}(\pi_1) \subseteq \text{supp}(\pi_2)$. Let $s = (s_1, \cdots, s_k)$ be a nontrivial part of π_1. Then, for $1 \leq i \leq k-1$, $\beta_i = [(s_i, s_{i+1})] \in \text{supp}(\pi)$. Thus, $\beta_i \in \text{supp}(\pi_2)$. Thus, there exists a part in π_2 containing the subparts (s_1, s_2) and (s_2, s_3). Since (s_1, s_2) and (s_2, s_3) both contain s_2, they must belong to the same part, or else π_2 would have two parts with the same number in it. Also, since s_2 can only appear once in any part, this part must contain (s_1, s_2, s_3). Continuing on in this manner, this part will contain $(s_1, \cdots, s_k)$. Thus, all nontrivial parts of π_1 are subparts of π_2. so $\pi_1 \leq \pi_2$. □

Lemma 2.2. If $\text{supp}(\pi_1) = \text{supp}(\pi_2)$, then $\pi_1 = \pi_2$.

Proof. Let s be a part of π_1. By the previous lemma, $\pi_1 \leq \pi_2$ so s is contained in a part t of π_2. Similarly, $\pi_2 \leq \pi_1$ so t is contained in a part of π_1. Since s and t share numbers, the part of π_1 which contains t must be s. Hence, $s = t$ and so every part of π_1 is a part of π_2. Analogous reasoning shows that every part of π_2 is a part of π_1. Thus, $\pi_1 = \pi_2$. □

Lemma 2.3. If $\pi_1 \in \mathcal{S}_n$ and $T \subseteq \text{supp}(\pi_1)$, then there exists a sequential partition π_2 such that $\text{supp}(\pi_2) = T$

Proof. Let $\beta = [(x,y)] \in \text{supp}(\pi_1) \setminus T$. Let $s = (s_1, \cdots, x, y, \cdots, s_k)$ be the part of π_1 containing (x,y). Let $u = (s_1, \cdots, x)$ and $v = (y, \cdots, s_k)$. Let π' be the sequential partition formed from π_1 by replacing the part s with the parts u and v. It is clear that $\beta \notin \text{supp}(\pi')$ and for every $\beta' \in \text{supp}(\pi_1), \beta' \neq \beta$, $\beta' \in \text{supp}(\pi')$. Thus, we can repeat this process to obtain a sequential partition containing all the elements of T but none of the elements of $\text{supp}(\pi_1) \setminus T$.

□

Theorem 2.4. For any $\pi \in \mathcal{S}_n$, $([\mathbf{0}, \pi], \leq)$ is isomorphic to $(\mathcal{P}(\text{supp}(\pi)), \subseteq)$.

Proof. This is a direct consequence of Lemmas 2.1, 2.2 and 2.3. □

These results allow us to define the join and meet of sequential partitions.

Definition 4. For any two compatible sequential partitions,$\pi_1, \pi_2 \in \mathcal{S}_n$, define the join of π_1 and π_2, denoted $\pi_1 \vee \pi_2$, as the element least sequential partition above both π_1 and π_2. That is to say $\pi_1 \vee \pi_2 = \pi$, where $\pi \in U(\pi_1) \cap U(\pi_2)$ and for all $\pi^* \in U(\pi_1) \cap U(\pi_2)$, $\pi \leq \pi^*$. If $\pi_1, \ldots, \pi_m$ is any sequence of sequential partitions, then $\bigvee_{i=1}^m \pi_i = \pi_1$ if $m = 1$ and $\pi_m \vee \left(\bigvee_{i=1}^{m-1} \pi_i\right)$ otherwise.

Notice that if π_1 and π_2 are incompatible, then $\pi_1 \vee \pi_2$ is undefined as $U(\pi_1) \cap U(\pi_2)$ is the empty set.

Definition 5. For any two sequential partitions (compatible or incompatible), $\pi_1, \pi_2 \in \mathcal{S}_n$, define the meet of π_1 and π_2, denoted $\pi_1 \wedge \pi_2$, as the greatest sequential partition below both π_1 and π_2. That is to say, $\pi_1 \wedge \pi_2 = \pi$, where $\pi \in D(\pi_1) \cap D(\pi_2)$ and for all $\pi^* \in P(\pi_1) \cap P(\pi_2)$, $\pi^* \leq \pi$. If $\pi_1, \cdots, \pi_m$ is an sequence of sequential partitions, then $\bigwedge_{i=1}^m \pi_i = \pi_1$if $m = 1$ and $\pi_m \wedge \left(\bigwedge_{i=1}^{m-1} \pi_i\right)$ otherwise.

Lemma 2.5. If $\Omega \in \mathcal{S}_n$, and $\pi_1, \pi_2 \leq \Omega$, then there exists a unique sequential partition $\pi \in [\mathbf{0}, \Omega]$ such that $\text{supp}(\pi) = \text{supp}(\pi_1) \cup \text{supp}(\pi_2)$, and $\pi = \pi_1 \vee \pi_2$. For any two sequential partitons π_1,π_2, then there exists a unique sequential partition π such that $\text{supp}(\pi) = \text{supp}(\pi_1) \cap \text{supp}(\pi_2)$ and $\pi = \pi_1 \wedge \pi_2$.

Proof. This is an obvious consequence of 2.4. □

Now for some notational conventions.

Definition 6. For any $\Omega \in \mathcal{S}_n$ and any $\pi \in [\mathbf{0}, \Omega]$ let $[\pi]_\Omega$ denote the set of permutations σ such that $\Omega \wedge \sigma = \pi$ and let $|\pi|_\Omega$ denote the number of permutations σ such that $\Omega \wedge \sigma = \pi$.

Definition 7. If Ω is any sequential partition and $S \subseteq [\mathbf{0}, \Omega]$ let $[S]_\Omega$ denote the set of permutations σ such that $\Omega \wedge \sigma \in S$ and let $|S|_\Omega$ denote the number of permutations, σ, such that $\Omega \wedge \sigma \in S$.

Definition 8. For any sequential partition $\pi = [s_1, \cdots, s_k]$ let $\nu(\pi) = k$, the number of parts in π. Let $\eta(\pi) = |\text{supp}(\pi)|$, the number of atomic partitions below π.

We close with an important definition of the *probability polynomial* of a set of sequential partitions and the *evaluator*, which we will see will be essential to computing $|S|_\Omega$ for a given set $S \subseteq [\mathbf{0}, \Omega]$.

Definition 9. Suppose we are given $\Omega \in \mathcal{S}_n$. Define $\mathcal{P}_\Omega : [\mathbf{0}, \Omega] \to Z[p]$ with,

$$\mathcal{P}_\Omega(\pi) = p^{\eta(\pi)}(1-p)^{\eta(\Omega)-\eta(\pi)}$$

.

If $S \subseteq [\mathbf{0}, \Omega]$ then the probability polynomial of S is ,

$$\mathcal{P}_\Omega(S) = \sum_{\pi \in S} \mathcal{P}_\Omega(\pi)$$

.

The evaluator is an operation on a polynomial in p defined as follows,

Definition 10. Given a polynomial in p, $F(p) = c_0 + \cdots c_k p^k$, and an integer $n > k$, the evaluator $[F]_n$ is

$$[F]_n = \sum_{j=0}^{k} c_j (n-j)!$$

.

Notice that if $\eta(\Omega) = n$ and $\eta(\pi) = k$, then $\mathcal{P}_\Omega(\pi)$ has the same form as the probability of getting a particular set of k heads (and no others) from n independent toss of coin when the probability of heads is p. This is not a coincidence. To derive the various values of $S^l_{r,d}(n,k)$ we will begin by first computing a polynomial $R(p)$ which represents the probability of a particular sequence of coin-tosses satisfies an analogous set of conditions. The computation of $[R(p)]_n$ converts evaluation of p^j to the evaluation of $(n-j)!$, which transforms the result from one about coin tosses to one about runs in permutations.

3 Some Useful Results

Lemma 3.1. Suppose $\Omega = [s_1, \cdots, s_k]$ is a sequential partition with k parts. For $1 \le i \le k$, let $\Pi_i = [s_i]$, the sequential partition whose only nontrivial part is s_i. Then, for each $\pi \le Omega$ there exists a unique set of sequential partitions $\pi_1, \cdots, \pi_k$ such that $\pi = \pi_1 \vee \cdots \vee \pi_k$ and $\pi_i \le \Pi_i$ for $1 \le i \le k$. Furthermore, every part of π is appears as a part of exactly one π_i.

Proof. Let $\pi_i = \Pi_i \wedge \pi$. Then, obviously $\pi_i \le \Pi_i$ for $1 \le i \le k$. Suppose s is a part of π. Since $\pi \le \Omega$, s is a subpart of Π_j for some value of j. But then $[s] \le \pi$ and $[s] \le \Pi_j$, so then, since $\pi_j = \Pi_j \wedge \pi$, $[s] \le \pi_j$. Thus, s is a subpart of π_j. Since, s is a part of π and $\pi_j \le \pi$ it must be the case that s is a part of π_j. Thus, every part of π is a part of π_i for some i. Thus, $\pi \le \pi_1 \vee \cdots \vee \pi_k$. But since $\pi_i \le \pi$ for $1 \le i \le k$ we also have that $\pi_1 \vee \cdots \vee \pi_k \le \pi$. Thus, $\pi = \pi_1 \vee \cdots \vee \pi_k$.

Suppose $\pi_1, \ldots, \pi_k$ form a set of sequential partitions satisfying the conditions. For each i, j with $i \ne j$, $\Pi_i \wedge \Pi_j = \mathbf{0}$. Since $\pi_i \le \Pi_i$, we have that if $i \ne j$ that $\operatorname{supp}(\pi_i) \cap \operatorname{supp}(\Pi_j) \subseteq \operatorname{supp}(\Pi_i) \cap \operatorname{supp}(\Pi_j) = \emptyset$. Thus, $\pi_i \wedge \Pi_j = \mathbf{0}$. It follows that

$$\begin{aligned}
\pi \wedge \Pi_j &= \left(\bigvee_{i=1}^{k} \pi_i \right) \wedge \Pi_j \\
&= \bigvee_{i=1}^{k} (\pi_i \wedge \Pi_j) \\
&= \mathbf{0} \vee \cdots \pi_j \cdots \vee \mathbf{0} \\
&= \pi_j
\end{aligned}$$

.

Thus, $\pi_j = \pi \wedge \Pi_j$ so the solution is unique. □

Lemma 3.2. For any sequential partition the number of permutations, σ, the number of permutations above π is $\nu(\pi)!$.

Proof. For any sequential partition π the set of permutations above π correspond to the possible orderings of the parts of π. There thus, $\nu(\pi)!$ permutations above π. □

Proposition 3.3. If $\pi \in \mathcal{S}_n$, then $n = \eta(\pi) + \nu(\pi)$.

Proof. Let the length of a part s of π be denoted by $|s|$. Note, that a part π corresponds to $|s| - 1$ elements of $\operatorname{supp}(\pi)$. Thus

$$\begin{aligned}
\eta(\pi) &= |\operatorname{supp}(\pi)| \\
&= \sum_{s \in \pi} (|s| - 1) \\
&= \sum_{s \in \pi} |s| - \sum_{s \in \pi} 1 \\
&= n - \nu(\pi)
\end{aligned}$$

□

Theorem 3.4. If $\Omega \in \mathcal{S}_n$ and if $\pi \in [\mathbf{0}, \Omega]$. Then, $|\pi|_\Omega$ is

$$\sum_{j=0}^{\eta(\Omega)-\eta(\pi)} (-1)^j \binom{\eta(\Omega) - \eta(\pi)}{j} (n - \eta(\pi) - j)! \tag{3.1}$$

.

Proof. Let S be the set of permutations above π and let $h = \eta(\Omega) - \eta(\pi)$. Let β_i, $1 \le i \le h$ be the elements of $\mathrm{supp}(\Omega) \setminus \mathrm{supp}(\pi)$. Let E_i be the intersection of the set of permutations above β_i with S. For any subset M of $[h]$ let $T_M = \bigcap_{i \in M} E_i$. Hence $\sigma \in T_M$ if and only if σ is above π and β_i for $i \in M$, or in other words $\mathrm{supp}(\pi) \subseteq \sigma$ and $\mathrm{supp}(\beta_i) \subseteq \pi$ for $i \in M$.

Let π_M be the sequential partition such that $\mathrm{supp}(\pi_M) = \mathrm{supp}(\pi) \cup \{\beta_i : i \in M\}$. Hence $\sigma \in T_M$ if and only if σ is above π_M. By Lemma 3.2 $|T_M| = \nu(\pi_M)! = (n - \eta(\pi_M))! = (n - \eta(\pi) - |M|)!$. Thus, for a given π, $|T_M|$ depends only on $|M|$, so let $T'_i = |T_M|$ for some M where $|M| = i$.

Let $R = S \setminus \bigcup_{i=1}^{h} E_i$. Thus, R is the set of permutations σ such that $\mathrm{supp}(\pi) \subseteq \mathrm{supp}(\sigma)$ and $\mathrm{supp}(\sigma) \cap (\mathrm{supp}(\Omega) \setminus \mathrm{supp}(\pi)) = \emptyset$. For any $\sigma \in R$ it follows that

$$\begin{aligned}
\mathrm{supp}(\sigma) \cap \mathrm{supp}(\Omega) &= \mathrm{supp}(\sigma) \cap (\mathrm{supp}(\pi) \cup (\mathrm{supp}(\Omega) \setminus \mathrm{supp}(\pi))) \\
&= (\mathrm{supp}(\sigma) \cap \mathrm{supp}(\pi)) \cup (\mathrm{supp}(\sigma) \cap (\mathrm{supp}(\Omega) \setminus \mathrm{supp}(\pi)) \\
&= \mathrm{supp}(\pi)
\end{aligned}$$

.

Thus $\sigma \in R$ if and only if $\sigma \wedge \Omega = \pi$. Then,

$$\begin{aligned}
|R| &= \left| S \setminus \bigcup_{i=1}^{h} E_i \right| \\
&= |S| - \left| \bigcup_{i=1}^{h} E_i \right| \\
&= |S| - \sum_{j=1}^{h} (-1)^{j-1} \left(\sum_{M \subseteq [h], |M| = j} |T_M| \right) \\
&= |S| + \sum_{j=1}^{h} (-1)^{j} \binom{h}{j} T'_j \\
&= (n - \eta(\pi))! - \sum_{j=1}^{h} (-1)^{j} \binom{h}{j} (n - \eta(\pi) - j)! \\
&= \sum_{j=0}^{h} (-1)^{j} \binom{h}{j} (n - \eta(\pi) - j)!
\end{aligned}$$

as desired. □

Corollary 3.5. If $\Omega \in \mathcal{S}_n$ and if $\pi \in [\mathbf{0}, \Omega]$. Then, $|\pi|_\Omega$ is

$$\left[p^{\eta(\pi)} (1-p)^{\eta(\Omega) - \eta(\pi)} \right]_n$$

.

Proof. This follows from 3.4 upon expanding the polynomial and computing the evaluator. □

Corollary 3.6. If $S \subseteq [\mathbf{0}, \Omega]$, then

$$|S|_\Omega = [\mathcal{P}_\Omega(S)]_n$$

.

4 Main Results

We have built up the machinery sufficient to prove our first result. From now on let Ω_r denote the sequential partition formed from all the runs of rise r in $\mathcal{S}_n$. More explicitly, let $P_i = (i, i+r, i+2r, \ldots, i+\lfloor (n-i)/r \rfloor r)$ for $1 \le i \le r$. Then $\Omega_r = [P_1, \cdots, P_r]$. Thus, for any permutation $\sigma \in S_n$ if $\pi = \Omega_r \wedge \sigma$, then the runs of rise r in σ are the subparts of π.

Theorem 4.1. The number of permutations in S_n containing exactly k runs with a rise of r and a distance of 1, can be computed by

$$S^2_{r,1}(n,k) = \left[\binom{n-r}{k} p^k (1-p)^{n-r-k} \right]_n$$

.

Proof. Let A be the set of sequential partitions $\pi \in [\mathbf{0}, \Omega_r]$ such that $\eta(\pi) = k$. Since supp is a bijection between sequential partitions in $[\mathbf{0}, \Omega_r]$ and subsets of supp(Ω_r), there are exactly $\binom{n-r}{k}$ elements in A. Then, $|A|_{\Omega_r}$ counts the number of permutations with exactly k runs of rise r. Then 3.5 implies

$$\begin{aligned} |A|_{\Omega_r} &= \sum_{\pi \in A} |\pi|_\Omega \\ &= \binom{n-r}{k} \left[p^k (1-p)^{n-r-k} \right]_n \\ &= \left[\binom{n-r}{k} p^k (1-p)^{n-r-k} \right]_n \end{aligned}$$

. □

Using, the coin analogy once more, the polynomial used to calculate $S^2_{r,1}(n,k)$ takes the form of the probability of getting exactly k heads out of $n-r$ independent coin flips. The evaluator uses this polynomial to calculate the number of permutations with exactly k runs from the set of $n-r$ possible runs of rise r.

5 Case: $\ell \ge 2, k = 0, d = 1, r = 1$

We now derive the formula for the number of permutations which contain no 2-runs, with a distance of 1 and a rise of 1. We will extend these results to count permutations containing a given number of runs of a given length with an arbitrary rise and distance.

For any sequential partition π, $\pi \le \Omega_1$, we have supp$(\pi) \subseteq$ supp(Ω_1). Thus, we can represent supp(π) as a binary vector $b = (b_1, \cdots, b_{n-1})$ such that $b_i = 1$ if $[(i, i+1)] \in$ supp(π) and $b_i = 0$ otherwise. Let $B_(\pi)$ denote the binary string corresponding to π in this manner. So if $\sigma \in S_n$ and $\pi = \Omega_1 \wedge \sigma$, then then the l-runs in σ correspond to runs of $l-1$ consecutive 1's in $B_{\Omega_1}(\pi)$. Furthermore, the number of 1's in $B_(\pi)$ is $\eta(\pi)$. Let $b_{n,l}(k)$ stand for the number of binary sequences of length n with no runs of l consecutive 1's and containing exactly k 1's. Then, let $R_{n,l}(p)$ be defined as

$$R_{n,l}(p) = \sum_{k=0}^{n-1} b_{n-1,l-1}(k) p^k (1-p)^{n-1-k} \tag{5.1}$$

.

We can use this polynomial to count permutations that of length n containing l-runs with

$$S^\ell_{1,1}(n,0) = [R_{n,l}(p)]_n \tag{5.2}$$

.

Thus, the problem is reduced to computing the probability polynomial $R_{n,l}(p)$. But in order to this, we must first be able to compute $b_{n,l}(k)$. For this, we employ the Goulden Jackson Cluster Method, described in the next section.

5.1 Goulden Jackson Cluster Method

Let $\mathbf{\Sigma}$ be an alphabet with d characters. A word is a sequence of elements from $\mathbf{\Sigma}$. Let $\mathbf{\Sigma}^*$ denote the set of all possible words (including the empty word with no characters). The length of a word is denoted by $|w|$. Given any word $w = w_1 \dots w_n$ a *factor* is a word of the form $w_i w_{i+1} \dots w_{j-1} w_j$, with $1 \le i \le j \le n$. A *proper factor* of a word w is any factor other than w itself. Now associate to each symbol $a \in \mathbf{\Sigma}$ a variable q_a. For any $w \in \mathbf{\Sigma}^*$, $w = w_1 \dots w_n$, we define the weight function $\text{weight}(w) = \prod_{i=1}^n q_{w_i}$. For example, if $\mathbf{\Sigma}$ were the English alphabet, then $\text{weight}(HIPHOP) = q_H^2 q_P^2 q_I q_O$. In general, if $\mathbf{\Sigma}$ has the characters $c_1, \dots, c_d$ and w is any word in $\mathbf{\Sigma}^*$ then $\text{weight}(w) = q_{c_1}^{e_1} \cdots q_{c_d}^{e_d}$ where e_i is the number of occurrences of the character c_i in w.

Let D be the set of "bad" words to be avoided. The set D must be such that no element of D is a proper factor of any other element of D. For example, D cannot contain both AC and $ACDC$. It is possible, however, for words in D to overlap. For instance, if AB and BA were in D, then ABA contains two overlapping factors. If a factor of a word w is an element of D, it is called a *marked factor*. Let $[i_1, j_1], \cdots, [i_k, j_k]$ be the start and end positions of the k marked factors in some word w. Since no element of D is contained in another, we can assume that $j_1 < j_2 < \cdots < j_d$ and all the i_x are distinct. A word is called a cluster if every letter in the word is part of a marked factor and neighboring marked factors, ordering them by their end positions, overlap. For instance, suppose again that AB and BA are in D. Then, $ABAB$ is composed of factors which overlap, an AB followed by a BA followed by another AB. But, $ABBA$ is not a cluster since the factor AB does not overlap with BA, and BB is not an element of D.

Suppose $\mathbf{\Sigma}$ is a language and $c_1, \cdots, c_d$ are the characters of $\mathbf{\Sigma}$. Given any (possibly infinite) set of words $\mathcal{S} \subseteq \mathbf{\Sigma}^*$, for any d-tuple $\mathbf{e} = (e_1, \dots, e_d)$, let $a_{\mathbf{e}}$ be the number of elements in $\mathcal{S}$ containing exactly e_i instances of the character c_i. Define the enumerator polynomial of $\mathcal{S}$ as the polynomial F that is the sum of the weights of all the words in $\mathcal{S}$. It follows that

$$
\begin{aligned}
F_S &= \sum_{w \in \mathcal{S}} \text{weight}(w) \\
&= \sum_{\mathbf{e} \in \mathbb{N}^d} a_{\mathbf{e}} \prod_{i=1}^{d} q_{c_i}^{e_i}
\end{aligned}
$$

.

Note that F will have an infinite number of terms if and only if S is infinite. The enumerating polynomials will serve as a useful tool for efficiently counting permutations with certain kinds of runs. In our case, we let $\mathcal{S}$ be the set of binary strings which contain no pairs of consecutive 1's. Every such binary string corresponds to a sequential partition, $\pi \le \Omega_1$, that represents permutations with no 3-runs.

Let D be a set of words to be avoided. Let $\mathcal{L}_D$ be the set of words in $\mathbf{\Sigma}^*$ which contain no words in D as factors. Let $\mathcal{C}_D$ be the collection of all clusters in $\mathbf{\Sigma}^*$. Define G_D as the enumerating polynomial for $\mathcal{L}_D$ and H_D as the enumerating polynomial for the $\mathcal{C}_D$. The Goulden-Jackson cluster method provides us with a simple method computing the generating function for H_D and G_D. Letting $Q = q_{c_1} + \cdots + q_{c_d}$, It is shown in [2] that

$$
G_D = \frac{1}{1 - Q - H_D} \tag{5.3}
$$

.

What follows is a brief description of the method to compute $\mathcal{H}_D$. Proof of the correctness of this algorithm can be found in [2].

For any word $w = w_1 \cdots w_n$, let HEAD(w) be the set of all proper prefixes:

$$\text{HEAD}(w) = \{w_1, w_1w_2, \ldots, w_1w_2\cdots w_{n-1}\}$$

and let TAIL(w) denote the set of all proper suffixes.

$$\text{TAIL}(w) = \{w_n, w_{n-1}w_n, \ldots, w_2\cdots w_n\}$$

.

Given two words u and v, define the set $\text{OVERLAP}(u, v) = \text{TAIL}(u) \cap \text{HEAD}(v)$. For instance, $\text{OVERLAP}(ABCABC, BCABCA) = \{BCABC, BC\}$. Now, if $x \in \text{HEAD}(v)$ then we can write $v = xx'$, where x' is the word obtained from v by removing its head x. Denote x' by $v \setminus x$. For example $DRAGON \setminus DRAG = ON$. For any two words u, v, define

$$u : v = \sum_{x \in \text{OVERLAP}(u,v)} \text{weight}(v \setminus x)$$

.

For example,

$$ABCABC : BCABCA = q_A + q_A^2 q_B q_C$$

For any $y \in D$, let $\mathcal{C}_D[y]$ denote the set of clusters whose final marked factor is y. Let $C_D[y]$ be the enumerator polynomial for the set $\mathcal{C}_D[y]$. The results [2] show the polynomials $C_D[x]$ satisfy the following relationship. For every $y \in D$,

$$C_D[y] = -\text{weight}(y) - \sum_{x \in D} (x : y) \cdot C_D[x], \tag{5.4}$$

.

Moreover, since the set $\{\mathcal{C}_D[x] | x \in D\}$ forms a partition of $\mathcal{C}_D$ it follows that

$$C_D = \sum_{x \in D} C_D[x].$$

Thus, by solving the system of equations in Equation 5.4 we can compute H_D and G_D.

5.2 Computing $b_{n,l}(k)$

Let $\Sigma = \{0, 1\}$ so $d = 2$. Let o_m refer to the word of m consecutive 1's. Let $D = \{o_m\}$. Let q_0 and q_1 be the variables associated with 0 and 1, respectively.

In order to compute C_D we must solve the system of equations specified in Equation 5.4. In this case D, only has one element so there is only one equation to solve. First we compute $o_m : o_m$. Since o_m consists of m consecutive 1's, $\text{OVERLAP}(o, o)$ includes the sequences of j consecutive 1's for $1 \leq j \leq m$. Thus

$$o_m : o_m = \sum_{i=1}^{m-1} q_1^i = \frac{q_1^m - q_1}{q_1 - 1}$$

.

From equation 5.4, we have

$$C_D[o_m] = -q_1^m - \frac{q_1^m - q_1}{q_1 - 1} C_D[o_m]$$
$$C_D[o_m] = -\frac{(1 - q_1)q_1^m}{(1 - q_1^m)}$$

.

Since o_m is the only element of D, $C_D[o] = C_D$. Thus, using Equation 5.3 we obtain the following formula for G_D.

$$\begin{aligned} G_D &= \frac{1 - q_1^m}{1 - q_0 - q_1 + q_0 q_1^m} \\ &= \sum_{i,j \geq 0} g_{i,j} q_0^i q_1^j \end{aligned}$$

Note that $g_{i,j}$ is the number of binary sequences with exactly i 0's, and j 1's which contain no subsequences of $l-1$ consecutive 1's. Thus, computing $b_{n,m}(k)$, the number of binary sequences of length n with no subsequences of m consecutive 1's that contain exactly k 1's is the same as computing the coefficient $g_{n-k,k}$.

In computing G_D with $D = \{o_m\}$, suppose we made the following substitutions. Substitute $(1-p)u$ fof q_0 and pu for q_1. Thus, G_D becomes

$$\begin{aligned} G_D &= \frac{1 - (pu)^m}{1 - u + p^m u^{m+1} - p^{m+1} u^{m+1}} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} g_{n-k,k} p^k (1-p)^{n-k} u^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_{n,m}(k) p^k (1-p)^{n-k} \right) u^n \end{aligned}$$

Recall that $R_{n,\ell}(p)$ is the probability of no sequences of $\ell - 1$ consecutive 1's in a binary string of length $n-1$. Then, $R_{n,\ell}(p)$ is the coefficient of u^{n-1} in G_D when $m = \ell - 1$. Thus,

$$\frac{1 - (up)^{\ell-1}}{1 - u + p^{l-1} u^{\ell} - p^{\ell} u^{\ell}} = \sum_{n=1}^{\infty} R_{n,\ell}(p) u^{n-1} \tag{5.5}$$

Using equation 5.2, we can compute $S_{1,1}^{\ell}(n,0)$ with $S_{1,1}^{\ell}(n,0) = [R_{n,\ell}(p)]_n$.

5.3 Example

To compute the value of $S_{1,1}^3(8,0)$ (note $\ell = 3$) we have

$$G_D = \frac{pu + 1}{(p-1)pu^2 + (p-1)u + 1}.$$

Using Mathematica, we see that the coefficient of u^7 is

$$R_{8,3}(p) = 1 - 6p^2 + 5p^3 + 6p^4 - 9p^5 + 3p^6.$$

Using definition 10,

$$[R_{8,3}(p)]_8 = 1 - 6(8-2)! + 5(8-3 + 6(8-4)! - 9(8-5)! + 3(8-6)! = 36,696$$

Thus, there are 36,696 permutations of length 8 containing no 3-runs

6 Case: $\ell \geq 2, k > 0, d = 1, r = 1$

This process is almost identical to the case with $k = 0$. The only difference is that we must now compute the two variable polynomial $R_{n,\ell}(p,t)$ so that the coefficient of $t^j u^n$ is the probability that a exactly j sets of ℓ consecutive heads appear in a sequence of n coin flips.

Let $\mathbf{\Sigma} = \{0,1\}$. Let D be a set of words such that no element of D is a factor of another. For each word $w \in \mathbf{\Sigma}$ let $e_D(w)$ be the number of factors of w that are elements of D. Let G_D be the series

$$G_D = \sum_{w \in \mathbf{\Sigma}^*} t^{e_D(w)} \text{weight}(w)$$

Then, the results of [2] give us a method to calculate G. Let $\mathcal{C}_D[y]$ denote the set of clusters whose final marked factor is y. Then, we must solve a system of equations similar to equation 5.4. Using the polynomial $C_D[y]$ for $y \in D$, we solve

$$C_D[y] = (t-1)\text{weight}(y) - \sum_{x \in D} (x : y) \cdot C_D[x], \tag{6.1}$$

If $H_D = \sum_{y \in D} C_D[y]$, we have that G_D is given by

$$G_D = \frac{1}{1 - (q_1 + \ldots + q_d) - H_D} \tag{6.2}$$

.

If o_m is the only element of D then, we compute G_D. Once gain we substitute pu and $(1-p)u$ for q_1 and q_0 in G_D. Then, $R_{n,\ell}(p,t)$ becomes the coefficient of u^{n-1} in the expansion of G_D when $m = \ell - 1$. The end result is

$$\begin{aligned} G_D &= \frac{1 - ptu - p^m u^m + tp^m u^m}{1 - u - ptu - tp^m u^{m+1} + p^m u^{m+1} + ptu^2 - p^{m+1}u^{m+1} + tp^{m+1}u^{m+1}} \\ &= \sum_{n=1}^{\infty} R_{n,\ell}(p,t) u^{n-1} \end{aligned}$$

.

6.1 Example

Let's compute the distribution $S^3_{1,1}(8,k)$ for $0 \leq k \leq 5$ (as there can be no more than 5 runs of length 3 in a permutation of length 8). Using Mathematica, we see that the coefficient of u^7 is

$$\begin{aligned} R_{8,3}(p,t) =& (1 + 3p^6 - 9p^5 + 6p^4 + 5p^3 - 6p^2) + \left(2p^7 - 14p^6 + 24p^5 - 8p^4 - 10p^3 + 6p^2\right) t \\ &+ \left(-7p^7 + 22p^6 - 18p^5 - 2p^4 + 5p^3\right) t^2 + \left(8p^7 - 12p^6 + 4p^4\right) t^3 \\ &+ \left(-2p^7 - p^6 + 3p^5\right) t^4 + \left(2p^6 - 2p^7\right) t^5 \end{aligned}$$

Then, $[R_{8,3}(p,t)]$ is computed by making the substitution $p^k \to (8-k)!$. The result is

$$[R_{8,3}(p,t)] = 36,969 + 3046t + 481t^2 + 80t^3 + 14t^5 + 2t^5$$

.

The coefficient of t^k in this expression represents the number of permutations with exactly k 3-runs of rise 1 and distance 1.

7 Case: $\ell \geq 2, k = 0, d = 1, r \geq 1$

The case for larger values of r can be handled with minimal modifications. Let σ be a permutation. For $1 \leq i \leq r$, let $P_i = (i, i+r, \ldots, i+r\lfloor(n-i)/r)\rfloor$. Let $\Omega_r = [P_1, \ldots, P_r]$, the sequential partition containing all runs of rise r. Let $\Pi_i = [(P_i)]$, the sequential partition formed from the i-th part of Ω_r. If σ is a permutation, let $\pi = \Omega_r \wedge \sigma$. The runs of rise r in σ are the subparts of π. Let S be the set of sequential partitions below Ω_r containing no part of length ℓ or greater. A permutation σ contains no ℓ-runs of rise r if and only if $\Omega_r \wedge \sigma \in S$.

For each $\pi \in S$, Lemma 3.1 implies there exists a unique set of sequential partitions $\pi_1, \ldots, \pi_r$ such that $\pi = \pi_1 \vee \cdots \vee \pi_r$, $\pi_i \leq \Pi_i$. For $1 \leq i \leq r$, let S_i be the set of sequential partitions below Π_i which contain no parts of size ℓ or greater. Then, $\pi \in S$ if and only if $\pi_i \in S_i$ for $1 \leq i \leq r$. Let $\widehat{S} = \{\boldsymbol{\pi} = (\pi_1, \ldots, \pi_r) \mid \pi_i \in S_i\}$. Hence,

$$\begin{aligned}
\mathcal{P}_{\Omega_r}(S) &= \sum_{\pi \in S} \mathcal{P}_{\Omega_r}(\pi) \\
&= \sum_{\pi \in S} p^{\eta(\pi)}(1-p)^{\eta(\Omega_r)-\eta(\pi)} \\
&= \sum_{\boldsymbol{\pi} \in \widehat{S}} p^{\eta(\pi_1 \wedge \cdots \wedge \pi_r)}(1-p)^{\eta(\Pi_1 \wedge \cdots \Pi_r)-\eta(\pi_1 \wedge \cdots \wedge \pi_r)} \\
&= \sum_{\boldsymbol{\pi} \in \widehat{S}} p^{\eta(\pi_1)+\cdots+\eta(\pi_r)}(1-p)^{(\eta(\Pi_1)-\eta(\pi_1))+\cdots+(\eta(\Pi_r)-\eta(\pi_r))} \\
&= \sum_{\boldsymbol{\pi} \in \widehat{S}} \left(p^{\eta(\pi_1)}(1-p)^{\eta(\Pi_1)-\eta(\pi_1)}\right) \cdots \left(p^{\eta(\pi_r)}(1-p)^{\eta(\Pi_r)-\eta(\pi_r)}\right) \\
&= \left[\sum_{\pi_1 \in S_1} p^{\eta(\pi_1)}(1-p)^{\eta(\Pi_1)-\eta(\pi_1)}\right] \cdots \left[\sum_{\pi_r \in S_r} p^{\eta(\pi_r)}(1-p)^{\eta(\Pi_r)-\eta(\pi_r)}\right] \\
&= \mathcal{P}_{\Pi_1}(S_1) \cdots \mathcal{P}_{\Pi_r}(S_r)
\end{aligned}$$

.

But, for any given i, we can do what we did before and represent elements below Π_i as binary sequences. Since Π_i contains a single part of length $n_i = 1 + \lfloor(n-i)/r\rfloor$, S_i is then just the set of binary sequences of length n_i with no sets of $\ell - 1$ consecutive 1's. Thus, $\mathcal{P}_{\Pi_i}(S_i)$ is just $R_{n_i,\ell}(p)$, which can be computed from Equation 5.5. It then follows that $|S|_\Omega = [\mathcal{P}_{\Omega_r}(S)]_n = [R_{n_1,l}(p) \cdots R_{n_r,p}(p)]_n$. Thus,

Theorem 7.1. Let $n_i = 1 + \lfloor(n-i)/r\rfloor$. The number of permutations with no ℓ-runs of rise r at a distance of 1 is given by

$$S^{\ell}_{r,1}(n, 0) = \left[\prod_{i=1}^{r} R_{n_i,\ell}(p)\right]_n$$

.

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