Exploring Extreme Points and Related Properties of Tsirelson Space

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Michael Holt

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Thesis Advisor: Dr. Kevin Beanland
Second Reader: Dr. Nathan Feldman

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ABSTRACT

Tsirelson space was constructed in 1974 as the first example of a Banach space without an embedded c_0 or l_p space. In 1989, Casazza and Shura wrote a book Tsirelson's Space devoted to Tsirelson space and its many properties. In this thesis, we give two representations of Tsirelson space and give an exposition of many results found in the Casazza-Shura book. In the final chapters, we make two mathematical contributions. First, we give new examples of extreme points of the unit ball of Tsirelson space, which expands the list of known ones from the Casazza-Shura book. Secondly, we improve the bounds on $j(n)$, which roughly measures the complexity of norming a vector of length n. We give an $O(\log_2(n))$ lower bound and an $O(\sqrt{n})$ upper bound. Both of these results answer questions from Tsirelson's Space, the second of which improves upon the $O(n)$ upper bound given by Casazza and Shura. This thesis is written to be accessible to mathematicians outside of functional analysis.

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CONTENTS

1

INTRODUCTION

In 1920, Stefan Banach highlighted and studied the classes of complete normed linear spaces, now known as Banach spaces [2]. While complete normed linear spaces were studied prior to Banach's work, his focus on them as a mathematical object was central to many later developments in analysis. Banach was mainly concerned with studying operators on and between Banach spaces, but eventually mathematicians began to study the isomorphic theory of Banach spaces and ask questions about it. As mentioned in [5], the main questions asked about an arbitrary Banach space X revolved around finding "nice" subspaces like a c_0 or an l_p, finding non-obvious examples of operators on X, and examining the non-obvious aspects of X's structure as a result of these non-obvious operators. As Banach spaces continued to gain popularity within the functional analysis community, many central questions in the isomorphic theory of Banach spaces remained open for decades. In the early 1990s, W.T. Gowers did pioneering work in Banach space theory by constructing many examples of infinite-dimensional Banach spaces which solved problems posed by Banach in the 1930s [5]. Gowers was eventually awarded the Field's Medal for this work in 1998. Much of Gowers' work owes its origins to work of Boris Tsirelson and his construction in 1974 of what is now called Tsirelson space.

Tsirelson space is considered the first example of a Banach space in which neither an l_p space nor a c_0 space can be embedded [1]. As a way to familiarize ourselves with and explore Tsirelson space, we relied heavily on the work of Peter Casazza and Thaddeus Shura in their book Tsirelson's Space. Published in 1989, this book serves as the backbone of our thesis [3]. Tsirelson's Space contains various theorems regarding properties of Tsirelson space, and we provide expositions of these proofs in hopes of clarifying them. Furthermore, the majority of our original work comes from trying to answer open questions within this text.

Tsirelson space is not a typical topic of conversation amongst undergraduate mathematics students, but we hope to make the topic accessible through this thesis. Our approach assumes that readers have a relatively limited knowledge of functional analysis and no knowledge of constructions in Banach space theory. Instead of focusing on the functional analysis side, most of our results are combinatorial in nature. To ease the reader into the topic, some basic concepts and notation of Banach spaces are laid out in Chapter 2. Then in the subsequent two chapters, we provide two different representations of Tsirelson space: one with norms and another with functionals. With this foundation in place, Chapter 5 serves to establish shortcuts for computing the Tsirelson norm and apply these shortcuts to prove results including how its unit vector basis is 1-unconditional.

Then we address two major problems in this thesis, the first of which involves finding extreme points of the unit ball of Tsirelson space. Thanks to Lindenstrauss' and Phelps' use of the Baire Category Theorem to develop a non-constructive proof that balls of reflexive Banach spaces have uncountably many extreme points, we know that Tsirelson space, a reflexive Banach space by definition, has uncountably many extreme points in its unit ball B_T [7]. In Tsirelson's Space, Casazza and Shura give examples of extreme points of B_T but do not give uncountably many [3]. Thus, we considered the problem of constructing uncountably many extreme points in B_T. Despite not solving this exact problem, our attempts to solve it led us to find methodologies for constructing new extreme points. Our new examples are presented in Chapter 6.

In the seventh chapter of the thesis, we consider another problem found in Tsirelson's Space. This book defines a quantity $j(n)$, which roughly measures the complexity of calculating the Tsirelson norm for vectors of length n. Although Casazza and Shura provide an upper bound for this quantity of $\lfloor (n-1)/2 \rfloor$, Problem 2(a) near the book's end asks for a tighter upper bound [3]. In an attempt to tighten this upper bound and the lower bound on $j(n)$, we give an $O(\sqrt{n})$ upper bound and an $O(\log_2(n))$ lower bound with the help of work in the thesis of Noah Duncan [4]. To wrap up the thesis, Chapter 8 contains the code used to help compute various norms. In Tsirelson's Space, they wrote code in Fortran to compute the norm values of vectors, while we use Python to do more than just compute a norm [3]. Yes, our Python programs include one to mimic the Casazza-Shura Fortran program by simply printing out the norm value given a vector and norm level, but we also provide all ways to norm the vector by showing how it can be broken up. In addition to these programs, we also wrote code to help determine extreme points of B_T and to enumerate all possible ways to usefully break a vector.

2

INTRODUCTION TO BANACH SPACES AND PRELIMINARIES IN NOTATION

Before defining Tsirelson space, we must lay the foundations of Banach spaces and introduce other significant definitions. We begin with the definitions of a norm on a real vector space and the corresponding normed linear space. The same definition holds for complex scalars but we restrict our attention to real scalars. The following definition will be necessary to define a Banach space.

Definition 1. *Suppose X is a real vector space. A norm $\|\cdot\|$ is a real-valued function satisfying the following three conditions:*

1. $\|x\| \geqslant 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = \vec{0}$,
2. *(Homogeneity)* $\|\lambda x\| = |\lambda|\|x\|$ *for all* $x \in X$, $\lambda \in \mathbb{R}$,
3. *(Triangle Inequality)* $\|x+y\| \leqslant \|x\| + \|y\|$ *for all* $x, y \in X$.

The pair $(X, \|\cdot\|)$, that is, the linear space X equipped with the norm $\|\cdot\|$, is called a normed linear space.

Note that the normed linear space $(X, \|\cdot\|)$ is complete provided all Cauchy sequences in X have limits in X. This precise definition of completeness is not important for our purposes.

Definition 2. *The normed linear space X is a Banach space provided X is complete with respect to its norm. Let $B_X = \{x \in X : \|x\| \leqslant 1\}$, the ball of X, and $S_X = \{x \in X : \|x\| = 1\}$, the sphere of X.*

Let $\mathbb{R}^\infty$ denote all sequences of real numbers. All Banach spaces we consider will be subspaces of $\mathbb{R}^\infty$. If a given $X \subseteq \mathbb{R}^\infty$ is an incomplete normed linear space, there there is a unique normed linear space $\hat{X} \subseteq \mathbb{R}^\infty$ such that $X \subseteq \hat{X}$, $\hat{X}$ is complete with respect to $X's$ norm, and X is dense in $\hat{X}$. The space $\hat{X}$ is a Banach space and is called the *completion* of X. As an easy example, the completion of $\mathbb{Q}$ with the absolute value norm is $\mathbb{R}$. The technical definition of completion uses equivalence classes of Cauchy sequences, so we will not give it here.

Let $x = (a_1, a_2, a_3, \ldots)$ be an infinite sequence of real numbers. The support of x is denoted supp $x = \{i : a_i \neq 0\}$. Then max supp x is the maximum element in supp x. Let c_{00} denote the vector space of all infinite sequences of real numbers whose support is finite. That is, each vector $x \in c_{00}$ is a finitely supported (eventually zero) sequence of real numbers.

Now we provide a couple examples of Banach spaces.

For $1 \leqslant p < \infty$, $(l_p, \|\cdot\|_p)$ is a Banach space, where

$$l_p = \left\{ (a_i)_{i=1}^{\infty} : \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{\frac{1}{p}} < \infty \right\}$$

and $\|(a_i)\|_p = (\sum_{i=1}^{\infty} |a_i|^p)^{\frac{1}{p}}$.

As another example, $(c_0, \|\cdot\|_0)$ is a Banach space, where

$$c_0 = \left\{ (a_i)_{i=1}^{\infty} : \lim_{i \to \infty} a_i = 0 \right\}$$

and $\|(a_i)\|_0 = \sup_{i \in \mathbb{N}} |a_i|$.

Although we have stated that $(l_p, \|\cdot\|_p)$ and $(c_0, \|\cdot\|_0)$ are Banach spaces, it is not trivial to show this. Conditions one and two from Definition 1 are easy to show, but the triangle inequality for $\|\cdot\|_p$ is called Minkowski's inequality. In addition it is cumbersome to prove that l_p and c_0 are complete with respect to their norms.

The approach we take in defining our Banach spaces is to define a norm on $c_{00} \subseteq \mathbb{R}^{\infty}$ and let the Banach space be the completion of c_{00} with respect to this norm. In this case, we lose touch with the exact description of the vectors in the resulting space. However, in most cases, this is not important as c_{00} is a dense subset of the resulting space. For example, taking the completion of c_{00} with respect to $\|\cdot\|_p$ gives us $(l_p, \|\cdot\|_p)$. Likewise, taking the completion of c_{00} with respect to $\|\cdot\|_0$ gives us $(c_0, \|\cdot\|_0)$.

For $x \in c_{00}$, let $x(i)$ be the i^{th} coordinate of x. The standard unit vectors $(e_i)_{i=1}^{\infty}$ of c_{00} are such that for each e_i, we have $e_i(i) = 1$ and $e_i(j) = 0$ for all $j \neq i$.

For all Banach spaces we are considering, particularly Tsirelson space, these vectors will form a *Schauder basis*. This means that the sequence $(a_i)_{i=1}^{\infty}$ is in the Banach space X if and only if $\sum_{i=1}^{\infty} a_i e_i$ is Cauchy with respect to the norm on X. This definition is not significant to our thesis but helps us name the sequence $(e_i)_{i=1}^{\infty}$ of standard unit vectors as the unit vector basis.

A last common notation is that any vector $x \in c_{00}$ can be written as $x = \sum_{i=1}^{\infty} x(i) e_i$, where $x(i)$ is the i^{th} coordinate of x. If $E \subset \mathbb{N}$ and $x \in c_{00}$ we define $Ex := \sum_{i \in E} x(i) e_i$.

3

TSIRELSON SPACE DEFINITION

In this chapter we provide the definition of Tsirelson space. To do so, we begin with a definition of a Schreier set, which will be used in the definition of the Tsirelson norm.

Definition 3. *A finite subset $F = \{n_1, n_2, \ldots, n_k\}$ of natural numbers where $n_1 < n_2 < \cdots < n_k$ is said to be a Schreier set, or admissible, if $k \leqslant n_1$. The set of all Schreier sets is denoted by S_1.*

For example, $\{2, 3\}$ is a Schreier set since $2 = k \leqslant n_1 = 2$. However, $\{3, 4, 5, 6\}$ is not a Schreier set since $4 = k > n_1 = 3$.

Definition 4. *Given two finite, non-empty sets E, F of natural numbers, we will write $E < F$ if $\max E < \min F$ and $E \leqslant F$ if $\max E \leqslant \min F$. For $k \in \mathbb{N}$ we write $k \leqslant E$ in place of $\{k\} \leqslant E$.*

For example, given $E = \{3, 4, 5\}$ and $F = \{7, 8, 9\}$, we say $E < F$ since $5 = \max E < \min F = 7$.

Next we define admissible sequences of subsets of natural numbers. This definition will place restrictions on how we can break up a given vector when taking the norm.

Definition 5. *For $k \in \mathbb{N}$, a sequence $(E_i)_{i=1}^k$ of finite, non-empty sets of natural numbers such that $k \leqslant E_1 < E_2 < \cdots < E_k$ is an admissible sequence.*

As an example, the sequence $(E_i)_{i=1}^3$ for $E_1 = \{3, 4, 5, 6\}$, $E_2 = \{7\}$, and $E_3 = \{10\}$ is an admissible sequence, since $k = 3 \leqslant E_1 < E_2 < E_3$. Note that the individual sets E_i need not be admissible.

3.1 UNDERSTANDING TSIRELSON SPACE

Tsirelson space is the completion of c_{00} with respect to a norm $\|\cdot\|_T$. The norm $\|\cdot\|_T$ is defined as the supremum over m of the increasing sequence of norms $(\|\cdot\|_m)_m$. These norms are defined recursively as follows: For $x \in c_{00}$

$$\|x\|_0 = \max_{n \in \mathbb{N}} |a_n| .$$

The completion of c_{00} with respect to this norm gives the Banach space c_0 which we mentioned in the introduction. Let us move onto the first level norm.

$$\|x\|_1 = \|x\|_0 \vee \sup\left\{\frac{1}{2}\sum_{j=1}^{k}\|E_jx\|_0 : k \in \mathbb{N},\ (E_j)_{j=1}^k \text{ admissible}\right\}.$$

To better understand the above definition, we consider a vector whose level zero and level one norms are different. For $x = (1,1,1,1,1,0,0,\ldots)$, we have that $\|x\|_0 = 1$ but

$$\|x\|_1 \geqslant \frac{1}{2}\sum_{j=1}^{3}\|E_jx\|_0 = \frac{1}{2}(\|E_1x\|_0 + \|E_2x\|_0 + \|E_3x\|_0) = \frac{1}{2}(1+1+1) = \frac{3}{2}$$

for $E_1 = \{3\}$, $E_2 = \{4\}$, and $E_3 = \{5\}$. In fact, $\|x\|_1 = \frac{3}{2}$.
Define the level two norm in a similar way.

$$\|x\|_2 = \|x\|_1 \vee \sup\left\{\frac{1}{2}\sum_{j=1}^{k}\|E_jx\|_1 : k \in \mathbb{N},\ (E_j)_{j=1}^k \text{ admissible}\right\}.$$

To understand this definition, we consider an example in which a vector has different level one and level two norms. Consider the vector $x = (1,1,1,1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,\ldots)$. We have

$$\|x\|_1 \geqslant \frac{1}{2}\sum_{j=1}^{3}\|E_jx\|_0 = \frac{1}{2}(\|E_1x\|_0 + \|E_2x\|_0 + \|E_3x\|_0) = \frac{1}{2}(1+1+\frac{1}{2}) = \frac{5}{4}$$

for $E_1 = \{3\}$, $E_2 = \{4\}$, and $E_3 = \{5,6,7,8\}$, while

$$\begin{aligned}\|x\|_2 \geqslant \frac{1}{2}\sum_{j=1}^{3}\|E_jx\|_1 &= \frac{1}{2}(\|E_1x\|_1 + \|E_2x\|_1 + \|E_3x\|_1)\\ &= \frac{1}{2}\left(1+1+\frac{1}{2}(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2})\right) = \frac{3}{2}\end{aligned}$$

for the same E_1, E_2, E_3 as above. It turns out that $\|x\|_1 = \frac{5}{4}$ and $\|x\|_2 = \frac{3}{2}$. We will continue in this pattern as we define the next level, which ultimately results in

$$\|x\|_{m+1} = \|x\|_m \vee \sup\left\{\frac{1}{2}\sum_{j=1}^{k}\|E_jx\|_m : k \in \mathbb{N},\ (E_j)_{j=1}^k \text{ admissible}\right\}$$

for any $m \geqslant 0$. Notice that by definition, for each $x \in c_{00}$ and $m \in \mathbb{N}$ we have $\|x\|_m \leqslant \|x\|_{m+1}$. Also note that for each $m \in \mathbb{N}$ we have $\|x\|_m \leqslant \sum\limits_{i=1}^{\infty}|x(i)| < \infty$ (since $x \in c_{00}$). Therefore for each $x \in c_{00}$ the sequence $(\|x\|_m)_m$ is increasing and bounded above. Therefore

$$\|x\|_T = \lim_m \|x\|_m$$

and is well defined. We collect all of these definitions and observations in the following.

Definition 6. *We define a sequence of norms* $(\|\cdot\|_m)_{m=0}^{\infty}$ *on* c_{00} *inductively. Let* $x = (a_i) \in c_{00}$ *and define* $\|\cdot\|_0$ *by*

$$\|x\|_0 = \max_{n \in \mathbb{N}} |a_n| \,, \tag{1}$$

and, assuming $\|\cdot\|_m$ *has been defined for* $m \geqslant 0$*, define* $\|\cdot\|_{m+1}$ *by*

$$\|x\|_{m+1} = \|x\|_m \vee \sup\left\{ \frac{1}{2} \sum_{j=1}^{k} \|E_j x\|_m : k \in \mathbb{N},\ (E_j)_{j=1}^{k} \text{ admissible} \right\}. \tag{2}$$

With the above sequence of norms defined on c_{00} we now introduce the Tsirelson norm.

Definition 7. *Given a vector* $x \in c_{00}$*, we define the Tsirelson norm of* x*, or* $\|x\|_T$*, as follows:*

$$\|x\|_T = \lim_{n \to \infty} \|x\|_n = \sup_{n \in \mathbb{N}} \|x\|_n. \tag{3}$$

Finally, we give the formal definition of Tsirelson space.

Definition 8. *Tsirelson space, or* T*, is the completion of* c_{00} *with respect to* $\|\cdot\|_T$*.*

4

NORMING SETS

This chapter offers an alternate presentation of the Tsirelson norm based on *norming sets*. These norming sets allow us to define norms on c_{00} like those in the previous chapter. While this representation of norms via norming sets is not significant in the majority of the thesis, we will frequently refer to the norming set for Tsirelson's space in Chapter 7. We begin by giving the definition of a norming set and showing how a norming set induces a norm on c_{00}. We then give some examples of norming sets which coincide with norms defined in Chapter 2. To conclude this chapter, we define a norming set for T.

Definition 9. *A set $W \subseteq c_{00}$ is a norming set if and only if the following conditions are satisfied.*

1. *$e_i^* \in W$ for all $i \in \mathbb{N}$, where each e_i^* is defined as e_i within the unit vector basis defined above,*
2. *$-f \in W$ for all $f \in W$.*

For a norming set W, an element $f \in W$ is called a functional. Using the above definitions, we prove the following proposition.

Proposition 10. *Let $W \subseteq c_{00}$ be a norming set. Then the function $\| \cdot \|_W : c_{00} \to [0, \infty)$ defined by*

$$\|x\|_W = \sup\{f(x) : f \in W\} \tag{4}$$

is a norm on c_{00}, where $f(x) := \langle f, x \rangle$ is the dot product of f and x.

The proof of the above proposition is usually omitted in the literature, but we include it here for the sake of completeness.

Proof. Let $W \subseteq c_{00}$ be a norming set. We must prove that all conditions within Definition 1 hold. Let $x \in c_{00}$. First we show that $\|x\|_W \geqslant 0$.

$$\|x\|_W \geqslant \langle sign(x(1))e_1^*, x \rangle = |x(1)| \geqslant 0,$$

where the sign function $sign$ returns 1 if its input is positive and -1 otherwise. The first inequality holds by definition, and the rest follow from there. So, $\|x\|_W \geqslant 0$, as desired.

Next we show that $\|x\|_W = 0$ if and only if $x = \vec{0}$. To prove the forward direction, assume $\|x\|_W = 0$. Then, since $e_i^* \in W$ for all $i \in \mathbb{N}$ by assumption, it must be that $e_i^*(x) = x(i) = 0$ for all $i \in \mathbb{N}$. This occurs only if $x = \vec{0}$, as desired.

To prove the reverse direction, assume $x = \vec{0}$. It trivially follows that $\|x\|_W = 0$, since the dot product of $\vec{0}$ and anything is still 0.

Now we prove that $\|\lambda x\|_W = |\lambda|\|x\|_W$ for all $\lambda \in \mathbb{R}$. Let $\lambda \in \mathbb{R}$. If $\lambda = 0$ we are done, as $\|\lambda x\|_W = 0 = |\lambda|\|x\|_W$. Suppose $\lambda \neq 0$. Then

$$\begin{aligned}
\|\lambda x\|_W &= \sup\{f(\lambda x) : f \in W\} \\
&= \sup\{|\lambda|(sign(\lambda)f)(x) : f \in W\} \\
&= |\lambda| \sup\{f(x) : f \in sign(\lambda)W\} \\
&= |\lambda| \sup\{f(x) : f \in W\} \\
&= |\lambda|\|x\|_W.
\end{aligned}$$

The first equality above follows from the definition of $\|\cdot\|_W$. The second and third equalities follows from the ability to rearrange scalars within dot products. The fourth equality comes from the fact that $W = -W$ where $-W = \{g : g = -f \text{ for some } f \in W\}$, based on condition (2) of our norming set definition. The last equality follows from our definition. Thus, $\|\lambda x\|_W = |\lambda|\|x\|_W$ for all $\lambda \in \mathbb{R}$, as desired.

Lastly, we must demonstrate that $\|x+y\|_W \leqslant \|x\|_W + \|y\|_W$. Fix $x, y \in c_{00}$. Let $g \in W$ and $\varepsilon > 0$ such that $\|x+y\|_W \leqslant g(x+y) + \varepsilon$ for some $g \in W$. So,

$$\|x+y\|_W \leqslant g(x+y) + \varepsilon = g(x) + g(y) + \varepsilon \leqslant \|x\|_W + \|y\|_W + \varepsilon.$$

The lone equality follows from rules of dot products, and the second inequality results from the definition of $\|\cdot\|_W$. So, since ε was arbitrary, the triangle inequality holds.

Therefore, $\|\cdot\|_W$ is a norm on c_{00} by Definition 1. $\square$

As an example of a norming set and its associated norm, consider the set

$$W_{c_0} = \{\pm e_i^* : i \in \mathbb{N}\}$$

and its associated norm

$$\|(a_i)\|_{W_{c_0}} = \sup_{i \in \mathbb{N}} |a_i| = \|(a_i)\|_0.$$

As another example, consider the norming set

$$W_{S_1} = \left\{ \sum_{i \in F} \pm e_i^* : F \in S_1 \right\}$$

and its associated norm

$$\|(a_i)\|_{W_{S_1}} = \sup_{F \in S_1} \sum_{i \in F} |a_i|.$$

The completetion of c_{00} with the above norm called Schreier's space and is an important precursor to Tsirelson's space.

As a last example, consider the norming set

$$W_{\ell_1} = \left\{ \sum_{i \in E} \pm e_i^* : E \subseteq \mathbb{N} \text{ is an interval} \right\}.$$

Then

$$\begin{aligned} \|x\|_{W_{\ell_1}} &= \sup\{f(x) : f \in W_{\ell_1}\} \\ &= \sup\left\{ \sum_{i \in E} \pm e_i^*(x) : E \subseteq \mathbb{N} \text{ is an interval} \right\} \\ &= \sum_{i \in \mathbb{N}} |x(i)| = \|x\|_{\ell_1}. \end{aligned}$$

Proposition 11. *Let $W \subseteq c_{00}$ be a norming set, and in addition assume $Ef \in W$ for any interval $E \subseteq \mathbb{N}$. Then for all intervals $E \subseteq \mathbb{N}$ and $x \in c_{00}$ we have $\|Ex\|_W \leqslant \|x\|_W$.*

Proof. Let $W \subseteq c_{00}$ as in the hypothesis, $x \in c_{00}$, and $E \subseteq \mathbb{N}$ be an interval. Let $\varepsilon > 0$ and find $f \in W$ such that $\|Ex\|_W \leqslant f(Ex) + \varepsilon$ for some $f \in W$. Then, as desired,

$$\|Ex\|_W \leqslant f(Ex) + \varepsilon = (Ef)(x) + \varepsilon \leqslant \|x\|_W + \varepsilon.$$

The lone equality above comes from the fact that E will restrict the range of our dot product in either place. So, since ε was arbitrary, we have $\|Ex\|_W \leqslant \|x\|_W$. □

We can now inductively define norming sets $W_1, W_2, W_3, \ldots$ so that W_n is the norming set for the norm $\|\cdot\|_n$ defined in Chapter 3 and $W = \cup_{n \in \mathbb{N}} W_n$ is the norming set for $\|\cdot\|_T$.

Definition 12. *Let*

$$W_0 = \{\pm e_i^* : i \in \mathbb{N}\}.$$

Let

$$W_1 = W_0 \cup \left\{ \frac{1}{2} \sum_{i \in F} \pm e_i^* : F \in S_1 \right\}.$$

For all $k \in \mathbb{N}$ with $k \geqslant 1$, let

$$W_{k+1} = W_k \cup \left\{ \frac{1}{2} \sum_{i=1}^{d} \pm E f_i : (\operatorname{supp} f_i)_{i=1}^d \text{ is admissible and } E \text{ is an interval} \right\}.$$

Then let $W = \cup_{k=1}^{\infty} W_k$.

Note that for all $m \in \mathbb{N} \cup \{0\}$, W_m satisfies all conditions necessary to be a norming set.

Proposition 13. *$\|\cdot\|_{W_k} = \|\cdot\|_k$ for all $k \in \mathbb{N} \cup \{0\}$.*

Proof. For $k \in \mathbb{N} \cup \{0\}$ let P_k be defined as follows: for all $x \in c_{00}$, $\|x\|_{W_k} = \|x\|_k$.
Base Case: As seen earlier in this chapter when we defined W_{c_0}, we see $W_{c_0} = W_0$ and we have already shown $\|(a_i)\|_{W_{c_0}} = \|(a_i)\|_0$.
Inductive Step: Now assume P_m holds for some $m \geqslant 0$. We now show that P_{m+1} holds as well. Let $x \in c_{00}$. By definition

$$\|x\|_{m+1} = \|x\|_m \vee \sup\left\{\frac{1}{2}\sum_{j=1}^{k} \|E_j x\|_m : k \in \mathbb{N},\ (E_j)_{j=1}^{k} \text{ admissible}\right\}.$$

If $\|x\|_{m+1} = \|x\|_m$, then by the Inductive Hypothesis

$$\|x\|_{m+1} = \|x\|_{W_m} \leqslant \|x\|_{W_{m+1}},$$

where the inequality holds since $W_m \subseteq W_{m+1}$.
Otherwise consider $(E_j)_{j=1}^{\infty}$ admissible.

$$\|x\|_{m+1} = \frac{1}{2}\sum_{j=1}^{d} \|E_j x\|_m = \frac{1}{2}\sum_{j=1}^{d} \|E_j x\|_{W_m}.$$

For each j find $f_j \in W_m$ so that $f_j(E_j x) = \|E_j x\|_{W_m}$. Let $g_j = E_j f_j \in W$. Since $(E_j)_{j=1}^{d}$ is admissible, $(\text{supp} g_j)_{j=1}^{\infty}$ is admissible. Thus,

$$\frac{1}{2}\sum_{j=1}^{d} \|E_j x\|_{W_m} = \frac{1}{2}\sum_{j=1}^{d} g_j(x) \leqslant \|x\|_{W_{m+1}}.$$

Thus $\|x\|_{m+1} \leqslant \|x\|_{W_{m+1}}$. Alternatively, find $f \in W_{m+1}$ such that $f(x) = \|x\|_{W_{m+1}}$. If $f \in W_m$ then $\|x\|_{W_{m+1}} = \|x\|_{W_m} = \|x\|_m \leqslant \|x\|_{m+1}$. If $f \in W_{m+1} \setminus W_m$ then $f = \frac{1}{2}(f_1 + \cdots + f_d)$ with $(\text{supp} f_i)_{i=1}^{d}$ admissible. So,

$$\|x\|_{W_{m+1}} = \frac{1}{2}\sum_{i=1}^{d} f_i(x) \leqslant \frac{1}{2}\sum_{i=1}^{d} \|E_i x\|_{W_m} = \frac{1}{2}\sum_{i=1}^{d} \|E_i x\|_m \leqslant \|x\|_{m+1},$$

where $E_i = range(f_i)$. Here $range(f_i)$ is the smallest interval containing f_i. $\square$

Based on the above proposition, we have that $\|\cdot\|_W = \|\cdot\|_T$ for $\|\cdot\|_T$ defined in Definition 7.

5

PROPERTIES OF TSIRELSON SPACE

This chapter serves to simplify our previous definition of the Tsirelson norm and to prove properties of Tsirelson space that result from these simplifications. Therefore the chapter contains two subsections: one to provide these alternate norm forms and the other to apply them.

5.1 ALTERNATE WAYS TO COMPUTE THE TSIRELSON NORM

Recall that for $x \in c_{00}$ we have $\|x\|_T = \sup_{m \in \mathbb{N}} \|x\|_m$ where for $m \geqslant 0$ the norm $\|\cdot\|_{m+1}$ satisfies:

$$\|x\|_{m+1} = \|x\|_m \vee \sup \left\{ \frac{1}{2} \sum_{j=1}^{k} \|E_j x\|_m : k \in \mathbb{N},\ (E_j)_{j=1}^{k} \text{ admissible} \right\}.$$

In subsequent chapters we will be computing the Tsirelson norm of various vectors, and we would like to do so as easily as possible. Therefore, in this chapter we will prove several results that provide shortcuts for calculating the norm of a given vector, which we now present as items in the below proposition.

Proposition 14. *Let $x \in c_{00}$. The following hold:*

1. *[3, Proposition I.10] For any $m \geqslant 0$*

$$\|x\|_{m+1} = \|x\|_0 \vee \sup \left\{ \frac{1}{2} \sum_{j=1}^{k} \|E_j x\|_m : k \in \mathbb{N}, (E_i)_{i=1}^{k} \text{ admissible} \right\}. \tag{5}$$

2. *[3, Ch.1 Remark 5] We have*

$$\|x\|_T = \|x\|_0 \vee \sup \left\{ \frac{1}{2} \sum_{i=1}^{k} \|E_i x\|_T : k \in \mathbb{N}, (E_i)_{i=1}^{k} \text{ admissible} \right\}. \tag{6}$$

3. *If* $\operatorname{minsupp} x = m$ *with* $m \geqslant 3$

$$\|x\|_T = \|x\|_0 \vee \sup \left\{ \frac{1}{2} \sum_{i=1}^{k} \|E_i x\|_T : k \geqslant m, (E_i)_{i=1}^{k} \text{ admissible} \right\}. \tag{7}$$

4. *We have*

$$\|x\|_T = \|x\|_0 \vee \sup\left\{\frac{1}{2}\sum_{i=1}^{k}\|E_i x\|_T : k \in \mathbb{N}, k \geqslant 3, (E_i)_{i=1}^{k} \text{ admissible}\right\}. \tag{8}$$

In the above proposition, Items 1 and 2 are results from [3], while Items 3 and 4 are original results. Although all four results seem similar, they tell us subtly different things. In particular, Item 1 switches out the first $\|x\|_m$ on the right-hand side of (2) for $\|x\|_0$, and this result will be used to prove $(e_i)_{i=1}^{\infty}$ is 1-unconditional, a term we will later define. Item 2 shows that the norm of T satisfies an implicit equation that is in terms of itself. This point is interesting because Tsirelson space is the first Banach space whose norm satisfies such an equation. Item 3 above shows that if a vector with $\operatorname{minsupp} x = m$ and $m \geqslant 3$ is normed by summing the norms of its various pieces, then the vector will at least be broken into m-many pieces. This result essentially says that a vector should always be broken into as many pieces as possible, which is slightly different from the other original result of Item 4. Item 4 says that we will never break a vector into less than three pieces. With the above proposition explained, we begin by proving its first item as seen in [3][pg. 13].

Proof. (Proposition 14)(Item 1) Let $x \in c_{00}$ and $m \geqslant 0$. From (2), we know

$$\|x\|_{m+1} = \|x\|_m \vee \sup\left\{\frac{1}{2}\sum_{j=1}^{k}\|E_j x\|_m : k \in \mathbb{N},\ (E_j)_{j=1}^{k} \text{ admissible}\right\},$$

so we just have to justify why

$$\|x\|_0 \vee \sup\left\{\frac{1}{2}\sum_{j=1}^{k}\|E_j x\|_m : k \in \mathbb{N},\ (E_j)_{j=1}^{k} \text{ admissible}\right\} =$$
$$\|x\|_m \vee \sup\left\{\frac{1}{2}\sum_{j=1}^{k}\|E_j x\|_m : k \in \mathbb{N},\ (E_j)_{j=1}^{k} \text{ admissible}\right\}.$$

Suppose

$$\|x\|_{m+1} > \sup\left\{\frac{1}{2}\sum_{j=1}^{k}\|E_j x\|_m : k \in \mathbb{N},\ (E_j)_{j=1}^{k} \text{ admissible}\right\}.$$

Then we know by definition that $\|x\|_{m+1} = \|x\|_m$. Therefore, we know

$$\begin{aligned}\|x\|_m &> \sup\left\{\frac{1}{2}\sum_{j=1}^{k}\|E_j x\|_m : k \in \mathbb{N},\ (E_j)_{j=1}^{k} \text{ admissible}\right\}\\ &\geqslant \sup\left\{\frac{1}{2}\sum_{j=1}^{k}\|E_j x\|_{m-1} : k \in \mathbb{N},\ (E_j)_{j=1}^{k} \text{ admissible}\right\},\end{aligned}$$

where the second inequality follows from the fact that $\|E_j x\|_m \geqslant \|E_j x\|_{m-1}$ for any E_j by definition. Hence $\|x\|_m = \|x\|_{m-1}$. Continuing in this manner, we conclude $\|x\|_{m+1} = \|x\|_0$. $\square$

Next, we prove the second item of Proposition 14, which offers a seemingly circular equation for the Tsirelson norm. We will prove it using the definition of Tsirelson norm and previous results in this thesis. This alternate representation provides a much simpler way to think about the norm.

Proof. (Proposition 14)(Item 2) Fix $x \in c_{00}$. Note that for all $(E_i)_{i=1}^k$ admissible and $m \geqslant 0$ we have

$$\|x\|_{m+1} \geqslant \frac{1}{2}\sum_{i=1}^{k} \|E_i x\|_m.$$

Taking $m \to \infty$ we have

$$\|x\|_T \geqslant \frac{1}{2}\sum_{i=1}^{k} \|E_i x\|_T.$$

Since $\|x\|_T \geqslant \|x\|_0$ we have

$$\|x\|_T \geqslant \max\left\{\|x\|_0, \sup\left\{\frac{1}{2}\sum_{i=1}^{k} \|E_i x\|_T\right\}\right\}.$$

In the other direction, fix $m \geqslant 0$ and find $(F_i)_{i=1}^k$ admissible with

$$\begin{aligned}\|x\|_{m+1} &= \frac{1}{2}\sum_{i=1}^{k} \|F_i x\|_m \\ &\leqslant \frac{1}{2}\sum_{i=1}^{k} \|F_i x\|_T \\ &\leqslant \max\left\{\|x\|_0, \sup\left\{\frac{1}{2}\sum_{i=1}^{k} \|E_i x\|_m\right\}\right\}.\end{aligned}$$

The first inequality above follows by definition of the Tsirelson norm, and the second inequality follows from Proposition 14 (Item 1). Taking $m \to \infty$ on the above inequality, we have

$$\|x\|_T \leqslant \max\left\{\|x\|_0, \sup\left\{\frac{1}{2}\sum_{i=1}^{k} \|E_i x\|_T\right\}\right\},$$

as desired. So,

$$\|x\|_T = \|x\|_0 \vee \sup\left\{\frac{1}{2}\sum_{i=1}^{k} \|E_i x\|_T : k \in \mathbb{N}, (E_i)_{i=1}^k \text{ admissible}\right\}.$$

This is the desired result. $\square$

Building on our above representation of the Tsirelson norm, we can specify the representation even more given the minimum of the support of the vector whose norm we are determining. In particular, we prove Item 3 of Proposition 14 in order to show that if a vector is normed by summing the norms of its various pieces, then the vector will be broken into as many pieces as possible.

Proof. (Proposition 14)(Item 3) Let $x \in c_{00}$ with $\operatorname{minsupp} x = m$ for $m \geqslant 3$. By (6), we have

$$\|x\|_T = \|x\|_0 \vee \sup \left\{ \frac{1}{2} \sum_{i=1}^{k} \|E_i x\|_T : k \in \mathbb{N}, (E_i)_{i=1}^{k} \text{ admissible} \right\},$$

so we just need to show that choosing an admissible $(E_i)_{i=1}^k$ with $k < m$ will not yield the supremum. Assume via contradiction that $\|x\|_T = \frac{1}{2} \sum\limits_{i=1}^{k} \|E_i x\|_T$ for some $(E_i)_{i=1}^k$ admissible and $k < m$. Since $\operatorname{minsupp} x = m$, we may assume that $\operatorname{minsupp} E_1 \geqslant m$, so we can have up to $(m-1)$-many sets after E_1 by definition of admissible. We can find some $j \in \{1, \dots, k\}$ such that $|E_j| \geqslant 2$ and write $E_j = E_{j,1} \cup E_{j,2}$ where $E_{j,1}$ and $E_{j,2}$ are intervals and the collection $\{E_1, \dots, E_{j,1}, E_{j,2}, \dots, E_k\}$ is admissible since $k < m$. Therefore if $k < m$ we can find a new admissible sequence which yields a greater norm than the original sequence, as we show below. Thus we may restrict the sequences $(E_i)_{i=1}^k$ to $k \geqslant m$.

Returning to our fixed $(E_i)_{i=1}^k$ with $k < m$, suppose without loss of generality that we must break up just E_1 into $E_{1,1}$ and $E_{1,2}$ in order for $E_{1,1} \cup E_{1,2} \cup (E_i)_{i=2}^k$ to contain m-many sets. Then, by the triangle inequality from Definition 1,

$$\frac{1}{2}(\|E_1 x\|_T + \|E_2 x\|_T + \cdots + \|E_k x\|_T) \leqslant \\ \frac{1}{2}(\|E_{1,1} x\|_T + \|E_{1,2} x\|_T + \|E_2 x\|_T + \cdots + \|E_k x\|_T),$$

so $E_{1,1} \cup E_{1,2} \cup (E_i)_{i=2}^k$ is an admissible set that yields the supremum. This contradicts $(E_i)_{i=1}^k$ yielding the supremum, so we are done. □

The above proof shows the importance of breaking a vector into the maximum number of pieces when summing the norms of its parts, and the below proof of Item 4 within Proposition 14 reveals that we will never break a vector into less than three pieces. This remark therefore builds on (6) to make it even stronger.

Proof. (Proposition 14)(Item 4) Note that the right hand side of (8) simply replaces $k \geqslant 1$ in (6) with $k \geqslant 3$. Therefore since we are taking the supremum over a smaller set, the inequality

$$\|x\|_T \geqslant \|x\|_0 \vee \sup \left\{ \frac{1}{2} \sum_{i=1}^{k} \|E_i x\|_T : k \in \mathbb{N}, k \geqslant 3, (E_i)_{i=1}^{k} \text{ admissible} \right\}$$

holds.

Let $x = \sum_i a_i e_i \in T$ and assume $\|x\|_0 < \|x\|_T$.

Suppose first that the supremum is attained for $k = 1$. Then $\|x\|_T = \frac{1}{2}\|E_1 x\|_T$ for some interval E_1. This cannot happen since $\|E_1 x\|_T \leqslant \|x\|_T$, which is a clear contradiction.

In the second case we show that if the supremum is attained for $k = 2$ then it also must be attained for some $k \geqslant 3$. If $\|x\|_T = \frac{1}{2}(\|E_1 x\|_T + \|E_2 x\|_T)$ then $\|x\|_T = \|E_1 x\|_T = \|E_2 x\|_T$ since $\|E_i x\|_T \leqslant \|x\|_T$ for all $i \in \mathbb{N}$. Note that $\min E_2 \geqslant 3$. Therefore

$$\begin{aligned}\|x\|_T = \|E_2 x\|_T &= \|E_2 x\|_0 \vee \sup\{\frac{1}{2}\sum_{i=1}^{k}\|F_i(E_2 x)\|_T : k \geqslant 3,\ (F_i)_{i=1}^k \text{ admissible}\} \\ &\leqslant \|x\|_0 \vee \sup\{\frac{1}{2}\sum_{i=1}^{k}\|F_i x\|_T : k \geqslant 3,\ (F_i)_{i=1}^k \text{ admissible}\} \leqslant \|x\|_T.\end{aligned}$$

We can restrict to $k \geqslant 3$ since $\min E_2 \geqslant 3$. Putting this together we have the desired result. □

5.2 APPLICATIONS OF PROPOSITION 14

With the help of the alternate ways to compute a norm found in Proposition 14, we can now prove several results. As a first application of this proposition, we show the basis (e_i) of T is 1-unconditional. We begin with the definition of 1-unconditional.

Definition 15. *Let $(x_n)_{n=1}^\infty$ be a sequence of vectors in a Banach space X. $(x_n)_{n=1}^\infty$ is 1-unconditional if for all $(a_i), (b_i) \in c_{00}$ with $|a_j| \leqslant |b_j|$ for $j \in \mathbb{N}$, we have $\|\sum_j a_j x_j\| \leqslant \|\sum_j b_j x_j\|$.*

From the above definition, we can easily see how $(e_i)_{i=1}^\infty$ is 1-unconditional for norms other than the Tsirelson norm. As an example, we consider the unit vector basis with respect to $\|\cdot\|_p$ for $1 \leqslant p < \infty$ below.

Remark 16. For $\|\cdot\|_p$ with $1 \leqslant p < \infty$, it is easy to see that the unit vector basis $(e_i)_{i=1}^\infty$ is 1-unconditional. Indeed, if we have $(a_i), (b_i) \in c_{00}$ with $|a_j| \leqslant |b_j|$ for $j \in \mathbb{N}$, then

$$\|\sum a_j e_j\|_p := \left(\sum_{i=1}^\infty |a_i|^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^\infty |b_i|^p\right)^{\frac{1}{p}} = \|\sum b_j e_j\|_p.$$

The below proposition will help us prove many other subsequent statements.

Proposition 17. *The unit vector basis $(e_i)_{i=1}^\infty$ is a 1-unconditional basis for T.*

Proof. We prove the above proposition by induction, using the following inductive statement P_n.

P_n : for all $(a_j)_{j=1}^\infty, (b_j)_{j=1}^\infty \in c_{00}$ such that $|a_j| \leqslant |b_j|$, for all $j \in \mathbb{N}$

$$\| \sum a_j e_j \|_n \leqslant \| \sum b_j e_j \|_n.$$

Base Case: Let $(a_j)_{j=1}^{\infty}$ and $(b_j)_{j=1}^{\infty}$ be scalar sequences such that $|a_j| \leqslant |b_j|$ for all $j \in \mathbb{N}$. For $n = 0$,

$$\| \sum a_j e_j \|_0 = \sup_{j \in \mathbb{N}} |a_j| \leqslant \sup_{j \in \mathbb{N}} |b_j| = \| \sum b_j e_j \|_0.$$

The two above equalities come from Definition 6, and the above inequality follows from the assumption that $|a_j| \leqslant |b_j|$ for all $j \in \mathbb{N}$. Thus, the base case holds.

Inductive Step: Now assume that P_n holds for some $n \in \mathbb{N}$. We show that P_{n+1} holds as well. Let $(a_j)_{j=1}^{\infty}$ and $(b_j)_{j=1}^{\infty}$ be scalar sequences such that $|a_j| \leqslant |b_j|$ for all $j \in \mathbb{N}$. By (5), we know there are just two options for $\| \sum a_j e_j \|_{n+1}$.

Case 1: Suppose $\| \sum a_j e_j \|_{n+1} = \| \sum a_j e_j \|_0$. Then we have

$$\left\| \sum a_j e_j \right\|_{n+1} = \left\| \sum a_j e_j \right\|_0 \leqslant \left\| \sum b_j e_j \right\|_0 \leqslant \left\| \sum b_j e_j \right\|_{n+1}.$$

The first inequality above follows from the fact that $|a_j| \leqslant |b_j|$ for all $j \in \mathbb{N}$, and the second inequality follows from Definition 6. Therefore $\| \sum a_j e_j \|_{n+1} \leqslant \| \sum b_j e_j \|_{n+1}$.

Case 2: Now suppose $\| \sum a_j e_j \|_{n+1} = \frac{1}{2} \sum_{i=1}^{k} \| E_i (\sum a_j e_j) \|_n$ for some $(E_i)_{i=1}^{k}$ admissible. So,

$$\| \sum a_j e_j \|_{n+1} = \frac{1}{2} \Big(\| \sum_{j \in E_1} a_j e_j \|_n + \| \sum_{j \in E_2} a_j e_j \|_n + \cdots + \| \sum_{j \in E_k} a_j e_j \|_n \Big).$$

Then, by our inductive hypothesis, we know that for each $m \in \{1, 2, \ldots, k\}$, $\| \sum_{j \in E_m} a_j e_j \|_n \leqslant \| \sum_{j \in E_m} b_j e_j \|_n$. Thus,

$$\begin{aligned}
\| \sum a_j e_j \|_{n+1} &= \frac{1}{2} \Big(\| \sum_{j \in E_1} a_j e_j \|_n + \| \sum_{j \in E_2} a_j e_j \|_n + \cdots + \| \sum_{j \in E_k} a_j e_j \|_n \Big) \\
&\leqslant \frac{1}{2} \Big(\| \sum_{j \in E_1} b_j e_j \|_n + \| \sum_{j \in E_2} b_j e_j \|_n + \cdots + \| \sum_{j \in E_k} b_j e_j \|_n \Big) \\
&= \frac{1}{2} \sum_{i=1}^{k} \| E_i (\sum b_j e_j) \|_n \leqslant \| \sum b_j e_j \|_{n+1}
\end{aligned}$$

The first inequality follows from the inductive hypothesis and the second inequality follows from (5). Therefore we have that P_{n+1} holds.

Since P_n holds for all $n \in \mathbb{N}$, let $(a_j)_{j=1}^{\infty}$ and $(b_j)_{j=1}^{\infty}$ be scalar sequences such that $|a_j| \leqslant |b_j|$ for all $j \in \mathbb{N}$. By Definition 7, $\| \sum a_j e_j \|_T = \sup_{n \in \mathbb{N}} \| \sum a_j e_j \|_n$. However, since $\sum a_j e_j \in c_{00}$,

$$\sup_{n \in \mathbb{N}} \| \sum a_j e_j \|_n = \max_{n \in \mathbb{N}} \| \sum a_j e_j \|_n = \| \sum a_j e_j \|_m$$

for some $m \in \mathbb{N}$. Then, we know

$$\|\sum a_j e_j\|_T = \|\sum a_j e_j\|_m \leqslant \|\sum b_j e_j\|_m \leqslant \|\sum b_j e_j\|_T.$$

The first inequality above comes from our P_n, and the second inequality above comes from Definition 7. Therefore, the unit vector basis $(e_i)_{i=1}^{\infty}$ of T is 1-unconditional, as desired. □

Now that we have proved $(e_i)_{i=1}^{\infty}$ is 1-unconditional, we can succinctly prove the following remark about the insignificance of the signs of the a_i's for a vector $(a_i) \in c_{00}$ when computing the Tsirelson norm.

Remark 18. Let $(a_i) \in c_{00}$ and $E \subset \mathbb{N}$ then

$$\|\sum_{i \in E} a_i e_i\|_T \leqslant \|\sum a_i e_i\|_T = \|\sum |a_i| e_i\|_T.$$

Proof. Both the inequality and equality follow from the unconditionality of the basis. For the inequality, we replace the coefficient a_i with 0 if and only if i is not in E. For the equality we note that the coefficient a_i and $|a_i|$ have the same absolute value. □

In addition to helping us prove that the unit vector basis is 1-unconditional, Proposition 14 can also be applied to give an upper bound on the Tsirelson norm. We now present this upper bound and prove it using equations from our proposition.

Lemma 19. *For all $(a_i) \in c_{00}$,*

$$\|\sum a_i e_i\|_T \leqslant \|\sum a_i e_i\|_0 \vee \frac{1}{2}\sum |a_i|.$$

Proof. Let $(a_i) \in c_{00}$. By (6), we know

$$\|\sum a_i e_i\|_T = \|\sum a_i e_i\|_0 \ \vee \ \sup\left\{\frac{1}{2}\sum_{j=1}^{k}\|E_j \sum a_i e_i\|_T : k \in \mathbb{N}, (E_j)_{i=1}^{k} \text{ admissible}\right\}.$$

Therefore, it suffices to show that for all $(E_j)_{j=1}^{k}$ admissible we have that

$$\frac{1}{2}\sum_{j=1}^{k}\|E_j \sum a_i e_i\|_T \leqslant \frac{1}{2}\sum |a_i|.$$

Let $(E_j)_{j=1}^k$ admissible for some $k \in \mathbb{N}$. Then

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{k}\|E_j \sum a_i e_i\|_T &\leqslant \frac{1}{2}\sum_{j=1}^{k}\sum_{i\in E_j}\|a_i e_i\|_T \\ &= \frac{1}{2}\sum_{j=1}^{k}\sum_{i\in E_j}|a_i|\,\|e_i\|_T \\ &= \frac{1}{2}\sum_{j=1}^{k}\sum_{i\in E_j}|a_i| \\ &\leqslant \frac{1}{2}\sum_{i=1}^{\infty}|a_i|.\end{aligned}$$

The first inequality follows from the triangle inequality applied to each $\|E_j \sum a_i e_i\|_T$. The first equality follows from the homogeneity of norms. The second equality follows from the fact that $\|e_i\|_T = 1$ for all $i \in \mathbb{N}$. The last inequality follows from the fact that for all $(E_j)_{j=1}^k$ admissible, $E_1 < E_2 < \cdots < E_k$, so we are adding up various pieces of the sequence $(a_i)_{i=1}^{\infty}$, not the whole sequence. $\square$

6

EXTREME POINTS OF B_T

This section is devoted to studying the extreme points of B_T. We begin with definition of extreme point of B_X.

Definition 20. *Let X be a Banach space. A vector $x \in B_X$ is an extreme point of B_X if there do not exist two vectors $x_1, x_2 \in B_X$ with $x_1, x_2 \neq x$ such that $x = \frac{1}{2}(x_1 + x_2)$. The set of all extreme points of B_X is denoted* ext*(B_X).*

We know there are uncountably many extreme points of B_T based on the following theorem in [7].

Theorem 21. *[7, Theorem 1.1 on p.39] If X is a reflexive Banach space, then the set* ext*(B_X) is uncountable.*

We will not recall the definition of reflexive Banach space here. The space T is a reflexive space. The standard way to see T is reflexive is to use a theorem of R.C. James [6] which states that a Banach space with an unconditional basis not containing isomorphic copies of c_0 or l_1 must be reflexive. As Tsirelson space was constructed as the first example of a space which does not contain c_0 or l_p for $1 \leqslant p < \infty$. In Proposition 17 we proved that the unit vector basis of T is 1-unconditional. In [7, Theorem 1.1 on p.39] Lindenstrauss and Phelps, use the Baire Category Theorem to show that ext(B_X) is not countable. As such they do not give a procedure for constructing uncountably many elements of ext(B_X). Thus it is desirable to find a way to exhibit uncountably many elements of ext(B_T) [3]. The first step in this process was given in [3, Lemma on p.202] where they show all vectors of the form

$$\varepsilon_1 e_1 + \varepsilon_2 e_2 + \varepsilon_i e_i + \varepsilon_j e_j$$

are extreme points of B_T, where $2 < i < j$ are in $\mathbb{N}$ and $\{\varepsilon_1, \varepsilon_2, \varepsilon_i, \varepsilon_j\} \in \{\pm 1\}$. We give self-contained exposition of this proof. In considering the question of constructing uncountably many elements of ext(B_T), we attempted to construct an element of ext(B_T) which was infinitely supported. Generating uncountably many elements of ext(B_T) would have followed from considering all sign changes of the coordinates of our infinitely many non-zero coordinates. Although we did not manage to construct such an infinitely supported element of ext(B_T), we were able to find new examples of extreme points in B_T, which we state in the following theorem. The theorem has four parts, the first of which comes from [3, Lemma on p.202] and the remaining of which are original results.

Theorem 22. *Let* $(\eta_i)_{i=1}^{\infty} \subset \{-1, 1\}$. *The following vectors are extreme points of* B_T*:*

1. *Vectors of the form*
$$\eta_1 e_1 + \eta_2 e_2 + \eta_i e_i + \eta_j e_j,$$
where $2 < i < j$ *are in* $\mathbb{N}$.
2. *Vectors of the form*
$$\eta_1 e_1 + \eta_2 e_2 + \frac{1}{2}\sum_{i=3}^{8} \eta_i e_i,$$
3. *Vectors of the form*
$$\eta_1 e_1 + \eta_2 e_2 + \frac{3}{5}\eta_3 e_3 + \frac{2}{5}\sum_{i=4}^{10} \eta_i e_i,$$
4. *Vectors of the form*
$$\eta_1 e_1 + \eta_2 e_2 + \frac{1}{2}\eta_3 e_3 + \frac{1}{3}\sum_{i=4}^{12} \eta_i e_i,$$

Notice that each of the above extreme points have ± 1 in the first two coordinates. This was observed in [3]. Before proving the above main theorem for this chapter, we will also prove this theorem.

Theorem 23. *[3, p.202] If* $x \in \mathrm{ext}(B_T)$ *then* $x(0), x(1) \in \{\pm 1\}$.

In order to prove the above theorems, we first need the assistance of some lemmas. Thus, we now prove the below lemma that any vector with only two non-zero coordinates, both of which are in $\{\pm 1\}$, has a Tsirelson norm of 1. This lemma is stated, but not proven, within [3, Lemma on p.202]. Therefore, we prove this lemma to add clarity to our exposition of their argument.

Lemma 24. *For* $i, j \in \mathbb{N}$ *and* $\{\varepsilon_i, \varepsilon_j\} \in \{\pm 1\}$, $\|\varepsilon_i e_i + \varepsilon_j e_j\|_T = 1$.

Proof. Let $i, j \in \mathbb{N}$ and let $\varepsilon_i, \varepsilon_j$ be signs. We show that $\|\varepsilon_i e_i + \varepsilon_j e_j\|_T = 1$. By Remark 18, we know that $\|\varepsilon_i e_i + \varepsilon_j e_j\|_T = \|e_i + e_j\|_T$. Clearly, since the zero norm minorizes the Tsirelson norm, $\|e_i + e_j\|_T \geqslant \|e_i + e_j\|_0 = 1$. Then by Lemma 19, we know
$$\|e_i + e_j\|_T \leqslant \|e_i + e_j\|_0 \vee \frac{1}{2}(1+1) = 1.$$
So, $1 \leqslant \|e_i + e_j\|_T \leqslant 1$, making $\|e_i + e_j\|_T$, and therefore $\|\varepsilon_i e_i + \varepsilon_j e_j\|_T$, equal to 1. □

Similarly, the below lemma will also be needed in our exposition of Theorem 22(Item 1). This lemma states that changing the first coordinate of a norm one vector to one will keep the vector norm one.

Lemma 25. *Let $(a_n) \in c_{00}$. Then if $\|\sum_{n=1}^{\infty} a_n e_n\|_T = 1$, we have*

$$\|e_1 + \sum_{n=2}^{\infty} a_n e_n\|_T = 1.$$

Proof. Let $\sum_{n=1}^{\infty} a_n e_n \in c_{00}$ be a vector of norm 1. First note that for all $n \in \mathbb{N}$, it must be that $|a_n| \leqslant 1$, as a coefficient with absolute value greater than 1 would make the zero norm, and therefore the Tsirelson norm, greater than 1. From (6), we have two options for $\|e_1 + \sum_{n=2}^{\infty} a_n e_n\|_T$. Note that $\|e_1 + \sum_{n=2}^{\infty} a_n e_n\|_0 = 1$, so if $\|e_1 + \sum_{n=2}^{\infty} a_n e_n\|_T = \|e_1 + \sum_{n=2}^{\infty} a_n e_n\|_0$ we are done.
Using (8) we suppose that for some $k \geqslant 3$ and for some $(E_m)_{m=1}^{k}$ admissible we have

$$\|e_1 + \sum_{n \geqslant 2} a_n e_n\|_T = \frac{1}{2} \sum_{m=1}^{k} \|E_m \sum a_n e_n\|_T.$$

However, by definition,

$$\frac{1}{2} \sum_{m=1}^{k} \|E_m \sum a_n e_n\|_T \leqslant \| \sum_{n=1}^{\infty} a_n e_n\|_T = 1.$$

This is the desired result, since we already know $\|e_1 + \sum_{n \geqslant 2} a_n e_n\|_T \geqslant 1$ by the zero norm. □

Based on the above two lemmas and the definition of an extreme point of B_T, we can now say that no norm 1 vector $\sum a_n e_n$ with $|a_1| < 1$ is an extreme point of B_T. This idea is still helping us incrementally build up to prove the main theorems of this chapter.

Lemma 26. *Any vector $\sum a_n e_n$ of norm 1 with $|a_1| < 1$ is not an extreme point of B_T.*

Proof. Let $\sum a_n e_n$ be a vector of norm 1 with $|a_1| < 1$. We will prove that $\sum a_n e_n$ is not an extreme point of B_T by finding two points x_1 and x_2 in B_T with $x_1, x_2 \neq x$ such that $\sum a_n e_n = \frac{1}{2}(x_1 + x_2)$.
Case 1: Suppose $a_1 \geqslant 0$. Then

$$\sum a_n e_n = \frac{1}{2}((1, a_2, a_3, \ldots) + (2a_1 - 1, a_2, a_3, \ldots)).$$

Since $\sum a_n e_n$ has norm 1, Lemma 25 gives that $(1, a_2, a_3, \ldots)$ is in B_T. Then, since $|2a_1 - 1| \leqslant 1$ and the unit vector basis is 1-unconditional, $\|(2a_1 - 1, a_2, a_3, \ldots)\|_T \leqslant \|(1, a_2, a_3, \ldots)\|_T = 1$, making $(2a_1 - 1, a_2, a_3, \ldots) \in B_T$ with a norm less than or equal to 1. Therefore, $\sum a_n e_n$ is not an

extreme point of B_T.
Case 2: Suppose $a_1 < 0$. By similar method to above, we see that

$$\sum a_n e_n = \frac{1}{2}((-1, a_2, a_3, \ldots) + (2a_1 + 1, a_2, a_3, \ldots)),$$

which makes $\sum a_n e_n$ not an extreme point of B_T, as desired. □

Now we introduce another lemma that says switching the first two coordinates of a vector will not change its Tsirelson norm. This idea is essential to prove Theorem 23.

Lemma 27. *Given a vector $x = \sum a_i e_i \in T$,*

$$\|\sum a_i e_i\|_T = \|a_2 e_1 + a_1 e_2 + \sum_{i=3}^{\infty} a_i e_i\|_T.$$

Proof. Let $x = \sum a_i e_i \in T$. Let $y = a_2 e_1 + a_1 e_2 + \sum_{i=3}^{\infty} a_i e_i$. We have $\|x\|_0 = \|y\|_0$. Note that $Ex = Ey$ whenever $\min E \geqslant 3$. Using (8) we have that

$$\begin{aligned} \|x\|_T &= \|x\|_0 \vee \sup\{\sum_{i=1}^{k} \|E_i x\|_T : k \geqslant 3,\ (E_i)_{i=1}^{k} \text{ is admissible}\} \\ &= \|y\|_0 \vee \sup\{\sum_{i=1}^{k} \|E_i y\|_T : k \geqslant 3,\ (E_i)_{i=1}^{k} \text{ is admissible}\} = \|y\|_T \end{aligned} \tag{9}$$

This is the desired result. □

With the above lemmas in place, we now provide our exposition of Theorem 23 [3, p.202].

Proof. (Theorem 23) Let $x = \sum_{i \in \mathbb{N}} a_i e_i$ with either $|a_1| < 1$ or $|a_2| < 1$. We show that $x \notin \text{ext}(B_T)$. If we can do so, then we are done by contrapositive. We know by Lemma 26 that if $|a_1| < 1$ then $x \notin \text{ext}(B_T)$. Thus, we just need to consider the case when $|a_2| < 1$. Assume via contradiction that $x \in \text{ext}(B_T)$ when we assume $|a_2| < 1$. Then by Lemma 26, it must be that $x = e_1 + \sum_{i \geqslant 2} a_i e_i$. By Lemma 27, we know that

$$\|e_1 + \sum_{i \geqslant 2} a_i e_i\|_T = \|a_2 e_1 + e_2 + \sum_{i \geqslant 3} a_i e_i\|_T.$$

.

We also know the above norms must equal one, since $x \in \text{ext}(B_T)$ by assumption and the zero norm is one. For convenience, denote $\sum_{i \geqslant 3} a_i e_i$ by x_3. So, $\|a_2 e_1 + e_2 + x_3\|_T = 1$. Then, using Lemma 25, we know

$$\|e_1 + e_2 + x_3\|_T = 1.$$

Since $|a_2| < 1$ by assumption, we can find an $\varepsilon > 0$ such that $|a_2 \pm \varepsilon| < 1$. Then we have

$$x = \frac{1}{2}\Big[\Big(e_1 + (a_2+\varepsilon)e_2 + x_3\Big) + \Big(e_1 + (a_2-\varepsilon)e_2 + x_3\Big)\Big].$$

Then, since Proposition 17 gives us that $(e_i)_{i=1}^{\infty}$ is 1-unconditional, we know

$$\|e_1 + (a_2+\varepsilon)e_2 + x_3\|_T \leqslant \|e_1 + e_2 + x_3\|_T = 1$$

and

$$\|e_1 + (a_2-\varepsilon)e_2 + x_3\|_T \leqslant \|e_1 + e_2 + x_3\|_T = 1.$$

Therefore by definition, $x \notin \text{ext}(B_T)$, which contradicts our assumption that $x \in \text{ext}(B_T)$. □

We now introduce a lemma that will be necessary to prove Theorem 22 (Item 1). This lemma shows that the signs of a vector's coordinates are not significant in determining whether or not it will be an extreme point.

Lemma 28. *Let $w = \sum a_i e_i$ and $x = \sum |a_i| e_i$. If $x \in \text{ext}(B_T)$, then $w \in \text{ext}(B_T)$.*

Proof. Let $w = \sum a_i e_i$ and $x = \sum |a_i| e_i$. Suppose there exist $w_1, w_2 \in B_T$ such that $w_1, w_2 \neq w$ and $w = \frac{1}{2}(w_1 + w_2)$. This is equivalent to supposing $w \notin \text{ext}(B_T)$. We will now show that this supposition implies $x \notin \text{ext}(B_T)$. To do so, we will define $x_1, x_2 \in B_T$ such that $x_1, x_2 \neq x$ and $x = \frac{1}{2}(x_1 + x_2)$. Let

$$x_k(i) = \begin{cases} w_k(i) & a_i \geqslant 0 \\ -w_k(i) & a_i < 0 \end{cases}$$

for $k \in \{1, 2\}$ and for all $i \in \mathbb{N}$. With this set-up, note that for all $i \in \mathbb{N}$ with $a_i \geqslant 0$,

$$\frac{1}{2}(x_1(i) + x_2(i)) = \frac{1}{2}(w_1(i) + w_2(i)) = w(i) = a_i = |a_i| = x(i),$$

and for all $i \in \mathbb{N}$ with $a_i < 0$,

$$\frac{1}{2}(x_1(i) + x_2(i)) = \frac{1}{2}(-w_1(i) + (-w_2(i))) = -w(i) = -a_i = |a_i| = x(i).$$

Then, $x = \frac{1}{2}(x_1 + x_2)$ for $x_1, x_2 \neq x$. Also, we know $\|x_k\|_T = \|w_k\|_T \leqslant 1$ for $k \in \{1, 2\}$, where the equality relies on Remark 18 and the inequality comes from the fact that $w_k \in B_T$. Therefore, $x \notin \text{ext}(B_T)$. □

Now we can more easily provide our exposition of the proof of Theorem 22 (Item 1).

Proof. (Theorem 22 (Item 1)) Let $i, j \in \mathbb{N}$ such that $2 < i < j$. Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_i, \varepsilon_j\} \in \{\pm 1\}$ and define $w = \varepsilon_1 e_1 + \varepsilon_2 e_2 + \varepsilon_i e_i + \varepsilon_j e_j$. We show $w \in \text{ext}(B_T)$. To do so, we begin by showing $w \in B_T$.

$$\begin{aligned}
\|w\|_T &= \|e_1 + e_2 + e_i + e_j\|_T \\
&= \max\{\|e_1 + e_2 + e_i + e_j\|_0, \|e_i + e_j\|_T\} \\
&= \max\{1, 1\} \\
&= 1.
\end{aligned}$$

The first equality above results from Remark 18. The second equality above uses (8) in combination with (6), while the third equality relies on Lemma 24. Therefore, we see that $w \in B_T$.

Now we will show $x = e_1 + e_2 + e_i + e_j$ is an extreme point of B_T. If we can do this, then we know w is also an extreme point of B_T by Lemma 28. Suppose $x = \frac{1}{2}(y + z)$ for $y, z \in B_T$. For $k \in \{1, 2, i, j\}$, we claim $y(k) = z(k) = 1$. Let $k \in \{1, 2, i, j\}$. If $y(k) < 1$, then $z(k) > 1$ in order to satisfy the equation $x(k) = \frac{1}{2}(y(k) + z(k))$. However, this makes $\|z\|_T \geqslant \|z\|_0 \geqslant |z(k)| > 1$, which contradicts $z \in B_T$. Therefore $y(k) \geqslant 1$, but $y(k) = 1$ or else we have the same contradiction. Since $y(k) = 1$, it must be that $z(k) = 1$ to make $x(k) = \frac{1}{2}(y(k) + z(k))$. So, $x(k) = y(k) = z(k) = 1$ for all $k \in \{1, 2, i, j\}$. Now we show that for all $\ell \in \mathbb{N} \setminus \{1, 2, i, j\}$ we have $y(\ell) = z(\ell) = x(\ell) = 0$. Fix $\ell \in \mathbb{N} \setminus \{1, 2, i, j\}$. We already know that $x(\ell) = 0$. Now assume via contradiction that $y(\ell) \neq 0$. We know $\ell \geqslant 3$, which makes $(E_r)_{r=1}^3$ admissible for $E_1 = \{\ell\}$, $E_2 = \{i\}$, and $E_3 = \{j\}$, supposing without loss of generality that $\ell < i$. Thus,

$$\begin{aligned}
\|y\|_T &\geqslant \frac{1}{2} \sup \left\{ \sum_{r=1}^{k} \|E_r y\|_T : k \in \mathbb{N}, (E_r)_{r=1}^k \text{ admissible} \right\} \\
&\geqslant \frac{1}{2} \Big(\|E_1 y\|_0 + \|E_2 y\|_0 + \|E_3 y\|_0 \Big) \\
&= \frac{1}{2}(2 + |y(\ell)|) \\
&= 1 + \frac{|y(\ell)|}{2} \\
&> 1.
\end{aligned}$$

The first inequality above comes from (6), and the rest of the above lines follow easily from there. So, $y \notin B_T$ if we assume $y(\ell) \neq 0$. This is a contradiction, so $y(\ell) = x(\ell) = 0$ for all $\ell \in \mathbb{N} \setminus \{1, 2, i, j\}$. The same method works to show $z(\ell) = x(\ell) = 0$ for all $\ell \in \mathbb{N} \setminus \{1, 2, i, j\}$, so $y = z = x$. Therefore, by Definition 20, x is an extreme point of B_T, which gives us that w, or $\varepsilon_1 e_1 + \varepsilon_2 e_2 + \varepsilon_i e_i + \varepsilon_j e_j$, is an extreme point of B_T as explained above. $\square$

Now that we have introduced some example extreme points of B_T from [3], we introduce some extreme points of B_T of our own by proving the other items within Theorem 22. To do so, we begin by proving the below lemma that gives the Tsirelson norm of a vector of e_i's.

Lemma 29. *Let $n \in \mathbb{N}$ with $n \geqslant 2$. Then*

$$\|\sum_{i=1}^{2n-1} e_i\|_T = \|\sum_{i=1}^{2n} e_i\|_T = \frac{n}{2}.$$

Proof. Let $n \in \mathbb{N}$ with $n \geqslant 2$. First we show $\|\sum_{i=1}^{2n-1} e_i\|_T = \frac{n}{2}$. Note that

$$\|\sum_{i=1}^{2n-1} e_i\|_T \geqslant \frac{1}{2}\sum_{i=n}^{2n-1} \|e_i\|_T = \frac{n}{2},$$

where the inequality comes from the fact that $\{n, n+1, \dots, 2n-1\}$ is a Schreier set and the equality follows from summing up n-many zero norms of 1.

Let $(E_i)_{i=1}^d$ be admissible. That is, $d \leqslant E_1 < E_2 < \cdots < E_d$ and each E_i is an interval. Let

$$D_1 = \Big\{ i \in \{1, 2, \dots, d\} : E_i \text{ is a singleton} \Big\}$$

and $D_2 = \{1, 2, \dots, d\} \setminus D_1$. Then, as desired,

$$\begin{aligned}
\|\sum_{i=1}^{2n-1} e_i\|_T &= \frac{1}{2}\sum_{i=1}^{d} \|E_i(\sum_{j=1}^{2n-1} e_j)\|_T \\
&= \frac{1}{2}\Big(|D_1| + \sum_{i \in D_2} \|\sum_{j \in E_i} e_j\|_T \Big) \\
&\leqslant \frac{1}{2}\Big(|D_1| + \sum_{i \in D_2} \frac{|E_i|}{2} \Big) \\
&= \frac{1}{2}\Big(|D_1| + \frac{1}{2}\sum_{i \in D_2} |E_i| \Big) \\
&\leqslant \frac{1}{2}\Big(|D_1| + \frac{1}{2}\Big((2n-1) - (d-1) - |D_1| \Big) \Big) \\
&= \frac{1}{2}\Big(|D_1| + n - \frac{1}{2} - \frac{d}{2} + \frac{1}{2} - \frac{|D_1|}{2} \Big) \\
&= \frac{1}{2}\Big(\frac{|D_1|}{2} - \frac{d}{2} + n \Big) \\
&\leqslant \frac{n}{2}.
\end{aligned}$$

Now we provide reasoning for the above relations. The first line follows from (6) and our knowledge that $\|\sum_{i=1}^{2n-1} e_i\|_T > \|\sum_{i=1}^{2n-1} e_i\|_0 = 1$. The second line follows from our definitions of D_1 and D_2. By Lemma 19, $\|\sum_{k \in E} e_k\|_T \leqslant \frac{|E|}{2}$, which explains the third line above. The fourth line comes from simplification of the previous line. To explain the fifth line, we observe that $|D_1| + |D_2| = d$ and

$$\left(\sum_{i \in D_2} |E_i|\right) + |D_1| \leqslant (2n-1) - (d-1),$$

making $\sum_{i \in D_2} |E_i| \leqslant (2n-1) - (d-1) - |D_1|$. Lines six and seven follow from simplifying expressions, while the last line relies on the fact that $|D_1| \leqslant d$. Therefore, $\|\sum_{i=1}^{2n-1} e_i\|_T = \frac{n}{2}$, as desired.

Now we show $\|\sum_{i=1}^{2n} e_i\|_T = \frac{n}{2}$. First note that the same methodology used above holds to show $\|\sum_{i=1}^{2n} e_i\|_T \geqslant \frac{n}{2}$. Now, we use the same $(E_i)_{i=1}^d$, D_1, and D_2 mentioned above to prove the other direction. Similar to above, we see that $\|\sum_{i=1}^{2n} e_i\|_T \leqslant \frac{1}{2}\left(|D_1| + \sum_{i \in D_2} \frac{|E_i|}{2}\right)$. If $|D_1| = d$ then $|D_2| = 0$, making $\sum_{i \in D_2} \frac{|E_i|}{2} = 0$. In this case, $\|\sum_{i=1}^{2n} e_i\|_T \leqslant \frac{1}{2}(|D_1|) = \frac{d}{2} \leqslant \frac{n}{2}$.

Now assume $|D_1| \leqslant d-1$. Then following the above methodology we get

$$\begin{aligned}
\|\sum_{i=1}^{2n} e_i\|_T &\leqslant \frac{1}{2}\left(|D_1| + \frac{1}{2}\Big((2n) - (d-1) - |D_1|\Big)\right) \\
&= \frac{1}{2}\left(|D_1| + n - \frac{d}{2} + \frac{1}{2} - \frac{|D_1|}{2}\right) \\
&= \frac{1}{2}\left(\frac{|D_1|}{2} - \frac{d}{2} + n + \frac{1}{2}\right) \\
&\leqslant \frac{1}{2}\left(\frac{d-1}{2} - \frac{d}{2} + n + \frac{1}{2}\right) \\
&= \frac{n}{2}.
\end{aligned}$$

Thus, $\|\sum_{i=1}^{2n} e_i\|_T = \|\sum_{i=1}^{2n-1} e_i\|_T = \frac{n}{2}$, as desired. $\square$

We will also need the below lemma involving matrices and linear algebra when proving that our vectors from Theorem 22 are extreme points.

Lemma 30. *For any $n \in \mathbb{N}$ with $n \geqslant 2$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{\pm 1\}$, the $n \times n$ matrix*

$$A = \begin{bmatrix} 0 & \varepsilon_2 & \varepsilon_3 & \ldots & \varepsilon_{n-1} & \varepsilon_n \\ \varepsilon_1 & 0 & \varepsilon_3 & \ldots & \varepsilon_{n-1} & \varepsilon_n \\ \varepsilon_1 & \varepsilon_2 & 0 & \ldots & \varepsilon_{n-1} & \varepsilon_n \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \ddots & \varepsilon_{n-1} & \varepsilon_n \\ \vdots & \vdots & \vdots & \ldots & 0 & \varepsilon_n \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \ldots & \varepsilon_{n-1} & 0 \end{bmatrix}$$

is invertible.

Proof. Let $n \in \mathbb{N}$ with $n \geqslant 2$ and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{\pm 1\}$. By factoring each ε_i out of the i^{th} column we have that $det(A) = \varepsilon_1\varepsilon_2 \cdots \varepsilon_n det(J_n - I_n)$ for J_n the $n \times n$ matrix of all 1's and I_n the $n \times n$ identity matrix. To show that $J_n - I_n$ has a non-zero determinant, we can equivalently show that it is invertible. Below we note that $(J_n - I_n)^{-1} = \frac{1}{n-1}J_n - I_n$:

$$\begin{aligned}
(J_n - I_n)(\frac{1}{n-1}J_n - I_n) &= \frac{1}{n-1}(J_n)^2 - (1 + \frac{1}{n-1})J_n + I_n \\
&= \frac{1}{n-1}(nJ_n) - (1 + \frac{1}{n-1})J_n + I_n \\
&= (-1 + \frac{n-1}{n-1})J_n + I_n \\
&= I_n.
\end{aligned}$$

All equalities above follow from simple matrix multiplication. In particular, the second equality follows from the fact that $(J_n)^2 = nJ_n$. Therefore, $J_n - I_n$ is invertible, making $J_n - I_n$ have a non-zero determinant. So, since $det(A) = \varepsilon_1\varepsilon_2 \cdots \varepsilon_n det(J_n - I_n)$ and each ε_i is also non-zero, A has a non-zero determinant and so is invertible. □

In particular, the above lemma will be used to prove the following one. We will see a similar pattern when proving that our extreme points are, in fact, extreme points. Thus, we capture that common pattern in the below lemma.

Lemma 31. *Let $x = \frac{2}{n}\sum_{i=n}^{2n} e_i$ for some $n \geqslant 2$. If there exist $y, z \in B_T$ such that $x = \frac{1}{2}(y+z)$, then $x(k) = y(k) = z(k)$ for all $k \geqslant n$.*

Proof. Let $n \geqslant 2$. Suppose $x = \frac{2}{n}\sum_{i=n}^{2n} e_i$ and we can find $y, z \in B_T$ such that $x = \frac{1}{2}(y+z)$. So, $y(k) = x(k) + \varepsilon_k\delta_k$ and $z(k) = x(k) - \varepsilon_k\delta_k$ for $k \in \mathbb{N}$, $\varepsilon_k \in \{\pm 1\}$, and $\delta_k \geqslant 0$. Observe that any $G \subseteq \{n, n+1, \ldots, 2n\}$ with $|G| = n$ is admissible. Therefore

$$\begin{aligned}
1 \geqslant \|y\|_T \geqslant \frac{1}{2}\sum_{k \in G} y(k) &= \frac{1}{2}\left((\sum_{k \in G} x(k)) + (\sum_{k \in G} \varepsilon_k\delta_k) \right) \\
&= \frac{1}{2}\left(n\left(\frac{2}{n}\right) + \sum_{k \in G} \varepsilon_k\delta_k \right) = 1 + \frac{1}{2}\sum_{k \in G} \varepsilon_k\delta_k.
\end{aligned}$$

The first inequality above results from $y \in B_T$, and the second inequality comes from the fact that G is a Schreier set. The rest follows easily, so it must be that $\sum_{k \in G} \varepsilon_k\delta_k \leqslant 0$. By similar argument on z, we see that $-\sum_{k \in G} \varepsilon_k\delta_k \leqslant 0$. Therefore it must be that $\sum_{k \in G} \varepsilon_k\delta_k = 0$. Since the above will be true for all possible G and there are $(n+1)$-many options for G, we have the following system of equations:

$$0\delta_n + \varepsilon_{n+1}\delta_{n+1} + \cdots + \varepsilon_{2n-1}\delta_{2n-1} + \varepsilon_{2n}\delta_{2n} = 0$$

$$\varepsilon_n\delta_n + 0\delta_{n+1} + \cdots + \varepsilon_{2n-1}\delta_{2n-1} + \varepsilon_{2n}\delta_{2n} = 0$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$\varepsilon_n\delta_n + \varepsilon_{n+1}\delta_{n+1} + \cdots + \varepsilon_{2n-1}\delta_{2n-1} + 0\delta_{2n} = 0.$$

Rewriting the above system as a matrix equation $A\mathbf{x} = 0$ yields:

$$\begin{bmatrix} 0 & \varepsilon_{n+1} & \varepsilon_{n+2} & \cdots & \varepsilon_{2n-1} & \varepsilon_{2n} \\ \varepsilon_n & 0 & \varepsilon_{n+2} & \cdots & \varepsilon_{2n-1} & \varepsilon_{2n} \\ \varepsilon_n & \varepsilon_{n+1} & 0 & \cdots & \varepsilon_{2n-1} & \varepsilon_{2n} \\ \varepsilon_n & \varepsilon_{n+1} & \varepsilon_{n+2} & \ddots & \varepsilon_{2n-1} & \varepsilon_{2n} \\ \vdots & \vdots & \vdots & \cdots & 0 & \varepsilon_{2n} \\ \varepsilon_n & \varepsilon_{n+1} & \varepsilon_{n+2} & \cdots & \varepsilon_{2n-1} & 0 \end{bmatrix} \begin{bmatrix} \delta_n \\ \delta_{n+1} \\ \delta_{n+2} \\ \vdots \\ \delta_{2n-1} \\ \delta_{2n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

Then, from Lemma 30, we know $\det(A) \neq 0$, which means the matrix equation $Ax = 0$ has only the trivial solution. Therefore we know $\delta_k = 0$ for all $k \in \{n, n+1, \ldots, 2n-1, 2n\}$. Hence, $x(k) = y(k) = z(k)$ for all $k \in \{n, n+1, \ldots, 2n-1, 2n\}$. Now consider $y(k) = \varepsilon_k\delta_k$ for some $k > 2n$, $\varepsilon_k \in \{\pm 1\}$, and $\delta_k \geqslant 0$. Note that $\{n+1, n+2, \ldots, 2n, k\}$ is a Schreier set for this choice of k. So, since y must be in B_T,

$$1 \geqslant \frac{1}{2}\left(\left(\sum_{j=n+1}^{2n} y(j)\right) + y(k)\right) = \frac{1}{2}\left(\left(n\left(\frac{2}{n}\right)\right) + y(k)\right) = 1 + \frac{1}{2}(y(k)).$$

Therefore $y(k) \leqslant 0$. So, $\varepsilon_k\delta_k \leqslant 0$. Using the same methodology for z, we get that $z(k) \leqslant 0$, so $-\varepsilon_k\delta_k \leqslant 0$. Therefore it must be that $\varepsilon_k\delta_k = 0$, showing $x(k) = y(k) = z(k) = 0$ for all $k > 2n$. Thus, $x(k) = y(k) = z(k)$ for all $k \geqslant n$. □

We are now able to prove the following new result, which is Theorem 22 (Item 2).

Proof. (Theorem 22 (Item 2)) Let $x = \eta_1 e_1 + \eta_2 e_2 + \frac{1}{2}\sum_{i\in F}\eta_i e_i$, for $F = \{3, 4, 5, 6, 7, 8\}$ and signs $\eta_i \in \{\pm 1\}$. Using Lemma 28 we can assume without loss of generality that $\eta_i = 1$ for $i \in F \cup \{1, 2\}$. First we show $\|x\|_T = 1$. Well, since $\|x\|_T \geqslant \|x\|_0 = 1$, we just need to show $\|x\|_T \leqslant 1$.

$$\begin{aligned} \|x\|_T &= \max\{\|x\|_0, \|\frac{1}{2}\sum_{i\in F} e_i\|_T\} \\ &= \max\{1, \frac{1}{2}\|\sum_{i\in F} e_i\|_T\} \\ &\leqslant \max\left\{1, \frac{1}{2}\left(\frac{4}{2}\right)\right\} \\ &= 1. \end{aligned}$$

The first equality above follows from (8). The other equalities follow from simple properties of norms. The lone inequality above results from Lemma 29.

Now suppose $x = \frac{1}{2}(y+z)$ for some $y, z \in B_T$. We will prove that $x = y = z$. By Lemma 31, we know $x(k) = y(k) = z(k)$ for all $k \geqslant 4$. Now consider $y(3) = x(3) + \varepsilon_3\delta_3$ and $z(3) = x(3) - \varepsilon_3\delta_3$ for some $\varepsilon_3 \in \{\pm 1\}$ and $\delta_3 \geqslant 0$. Now suppose via contradiction $\delta_3 > 0$. Then, supposing without loss of generality that $y(3) = \frac{1}{2} + \delta_3$, we see that the admissible sequence $(\{3\}, \{4\}, \{5,6,7,8\})$ yields

$$\|y\|_2 \geqslant \frac{1}{2}\left(\left(\frac{1}{2}+\delta_3\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)\right)\right) = \frac{1}{2}(2+\delta_3) > 1.$$

This contradicts the fact that $y \in B_T$. Therefore, $\delta_3 = 0$, and $x(3) = y(3) = z(3)$. Lastly, arguing as in the proof of Theorem 22 (Item 1), $x(1) = x(2) = y(1) = y(2) = z(1) = z(2) = 1$, since any perturbation of 1 will cause a zero norm to be more than 1. Therefore $x = y = z$, and x is an extreme point of B_T by Definition 20. $\square$

Now we follow the same methodology as above to prove Item 3 within Theorem 22.

Proof. (Theorem 22 (Item 3)) Let $F = \{4,5,6,7,8,9,10\}$ and let $x = \eta_1 e_1 + \eta_2 e_2 + \frac{3}{5}\eta_3 e_3 + \frac{2}{5}\sum_{i\in F}\eta_i e_i$ where each $\eta_i \in \{\pm 1\}$. Using Lemma 28 we can assume without loss of generality that $\eta_i = 1$ for $i \in F \cup \{1,2,3\}$. First we show $\|x\|_T = 1$. Well, since $\|x\|_T \geqslant \|x\|_0 = 1$, we just need to show $\|x\|_T \leqslant 1$.

By (8), if $\|x\|_T \neq \|x\|_0$, then $\|x\|_T = \sup\left\{\frac{1}{2}\sum\limits_{i=1}^{k}\|E_i x\|_T : k \in \mathbb{N}, k \geqslant 3, (E_i)_{i=1}^k \text{ admissible}\right\}$. So, for some $(E_i)_{i=1}^k$ admissible for $k \geqslant 3$, we have $\|x\|_T = \frac{1}{2}\sum_{i=1}^k \|E_i x\|_T$. Note that if $k \geqslant 4$, we are essentially working with a vector of all $\frac{2}{5}$'s with $\max\operatorname{supp} x = 10$ and Lemma 29 gives the norm of x as $\frac{2}{5}(\frac{5}{2})$, or 1, so we assume $k = 3$. Thus, $\|x\|_T = \frac{1}{2}\Big(\|E_1x\|_T + \|E_2x\|_T + \|E_3x\|_T\Big)$.

By Lemma 19, we know each $\|E_i x\|_T \leqslant \|E_i x\|_0 \vee \frac{1}{2}\sum_{j\in E_i}|x(j)|$, so we have two options for each $\|E_i x\|_T$.

Case 1: Suppose $\|E_i x\|_T \leqslant \frac{1}{2}\sum_{j\in E_i}|x(j)|$ for all $i \in \{1,2,3\}$. Then

$$\|x\|_T = \frac{1}{2}\Big(\|E_1x\|_T + \|E_2x\|_T + \|E_3x\|_T\Big) \leqslant \frac{1}{2}\left(\frac{1}{2}\left(\frac{3}{5}+\frac{14}{5}\right)\right) = \frac{17}{20} < 1.$$

Case 2: Suppose $\|E_i x\|_T \leqslant \frac{1}{2}\sum_{j\in E_i}|x(j)|$ for two $i \in \{1,2,3\}$.

Case 2(a): Suppose $\|E_1x\|_T \leqslant \|E_1x\|_0 = \frac{3}{5}$. Then $\|E_2x\|_T + \|E_3x\|_T \leqslant \frac{1}{2}(\frac{14}{5})$. So,

$$\|x\|_T = \frac{1}{2}\Big(\|E_1x\|_T + \|E_2x\|_T + \|E_3x\|_T\Big) \leqslant \frac{1}{2}\left(\frac{3}{5}+\frac{1}{2}\left(\frac{14}{5}\right)\right) = \frac{3}{10}+\frac{7}{10} = 1.$$

Case 2(b): Suppose $\|E_2x\|_T \leqslant \|E_2x\|_0 = \frac{2}{5}$. Then $\|E_1x\|_T + \|E_3x\|_T \leqslant \frac{1}{2}(\frac{3}{5} + \frac{12}{5})$. So,

$$\|x\|_T = \frac{1}{2}\Big(\|E_1x\|_T + \|E_2x\|_T + \|E_3x\|_T\Big) \leqslant \frac{1}{2}\left(\frac{2}{5} + \frac{1}{2}\left(\frac{3}{5} + \frac{12}{5}\right)\right) = \frac{1}{5} + \frac{3}{4} = \frac{19}{20} < 1.$$

Case 2(c): Suppose $\|E_3x\|_T \leqslant \|E_3x\|_0 = \frac{2}{5}$. The same argument used in Case 2(b) works to show $\|x\|_T \leqslant 1$ in this case.

Case 3: Suppose $\|E_ix\|_T \leqslant \frac{1}{2}\sum_{j\in E_i}|x(j)|$ for only one $i \in \{1,2,3\}$.

Case 3(a): Suppose $\|E_1x\|_T \leqslant \frac{1}{2}\sum_{j\in E_1}|x(j)|$. To get an upper bound, we suppose E_1x is a vector of $\frac{3}{5}$'s with $\max\operatorname{supp} E_1x = 8$ to get $\|E_1x\|_T \leqslant \frac{3}{5}(\frac{4}{2}) = \frac{6}{5}$. This upper bound holds by Lemma 29. Also, $\|E_2x\|_T = \|E_3x\|_T \leqslant \frac{2}{5}$, so

$$\|x\|_T = \frac{1}{2}\Big(\|E_1x\|_T + \|E_2x\|_T + \|E_3x\|_T\Big) \leqslant \frac{1}{2}\left(\frac{6}{5} + \frac{4}{5}\right) = \frac{1}{2}(2) = 1.$$

Case 3(b): Suppose $\|E_2x\|_T \leqslant \frac{1}{2}\sum_{j\in E_2}|x(j)|$. By Lemma 29, $\|E_2x\|_T \leqslant \frac{2}{5}(\frac{5}{2}) = 1$. Also, $\|E_1x\|_T \leqslant \frac{3}{5}$ and $\|E_3x\|_T \leqslant \frac{2}{5}$, so

$$\|x\|_T = \frac{1}{2}\Big(\|E_1x\|_T + \|E_2x\|_T + \|E_3x\|_T\Big) \leqslant \frac{1}{2}\left(\frac{3}{5} + 1 + \frac{2}{5}\right) = \frac{1}{2}(2) = 1.$$

Case 3(c): Suppose $\|E_3x\|_T \leqslant \frac{1}{2}\sum_{j\in E_3}|x(j)|$. The same argument used in Case 3(b) works to show $\|x\|_T \leqslant 1$ in this case.

Case 4: Suppose $\|E_ix\|_T \leqslant \|E_ix\|_0$ for all $i \in \{1,2,3\}$. Then

$$\|x\|_T = \frac{1}{2}\Big(\|E_1x\|_T + \|E_2x\|_T + \|E_3x\|_T\Big) \leqslant \frac{1}{2}\left(\frac{3}{5} + \frac{2}{5} + \frac{2}{5}\right) = \frac{1}{2}\left(\frac{7}{5}\right) = \frac{7}{10} < 1.$$

Therefore, $1 \leqslant \|x\|_T \leqslant 1$ in all cases, so $\|x\|_T = 1$.

Now suppose $x = \frac{1}{2}(y+z)$ for some $y, z \in B_T$. We will prove that $x = y = z$. Arguing as in the proof of Theorem 22 (Item 1), we know that $x(1) = x(2) = y(1) = y(2) = z(1) = z(2) = 1$. By Lemma 31, $x(k) = y(k) = z(k)$ for all $k \geqslant 5$. Next we want to show $x(4) = y(4) = z(4)$. We know $y(4) = \frac{2}{5} + \varepsilon_4\delta_4$ and $z(4) = \frac{2}{5} - \varepsilon_4\delta_4$ for some $\varepsilon_4 \in \{\pm 1\}$ and some $\delta_4 \geqslant 0$. Assume via contradiction $\delta_4 > 0$. Then, supposing without loss of generality that $y(4) = \frac{2}{5} + \delta_4$, we see that the admissible sequence $(\{4\}, \{5\}, \{6\}, \{7,8,9,10\})$ yields

$$\|y\|_2 \geqslant \frac{1}{2}\left(\left(\frac{2}{5} + \delta_4\right) + \left(\frac{2}{5}\right) + \left(\frac{2}{5}\right) + \left(\frac{1}{2}\left(\frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5}\right)\right)\right) = \frac{1}{2}(2 + \delta_4) > 1.$$

This contradicts the fact that $y \in B_T$. Therefore, $\delta_4 = 0$, as desired.

Now we want to show $x(3) = y(3) = z(3)$. We know $y(3) = \frac{3}{5} + \varepsilon_3\delta_3$ and $z(3) = \frac{3}{5} - \varepsilon_3\delta_3$ for some $\varepsilon_3 \in \{\pm 1\}$ and some $\delta_3 \geqslant 0$. Assume via contradiction $\delta_3 > 0$ and suppose without loss of generality that $y(3) = \frac{3}{5} + \delta_3$. Then the admissible sequence $(\{3\}, \{4\}, \{5,6,7,8,9\})$ yields

$$\|y\|_2 \geqslant \frac{1}{2}\left(\left(\frac{3}{5} + \delta_3\right) + \frac{2}{5} + \frac{1}{2}\left(5\left(\frac{2}{5}\right)\right)\right) = \frac{1}{2}(2 + \delta_3) > 1,$$

which contradicts $y \in B_T$. Thus, it must be that $\delta_3 = 0$. So, $y(3) = z(3) = \frac{3}{5}$.

Putting it all together we have $x = y = z$, as desired. Therefore x is an extreme point of B_T by Definition 20. $\square$

Once again, we will follow a very similar methodology to the one used above when proving the fourth item in our main theorem for this chapter.

Proof. (Theorem 22 (Item 4)) Let $F = \{4,5,6,7,8,9,10,11,12\}$ and let $x = \eta_1 e_1 + \eta_2 e_2 + \frac{1}{2}\eta_3 e_3 + \frac{1}{3}\sum_{i \in F} \eta_i e_i$ where each $\eta_i \in \{\pm 1\}$. Using Lemma 28 we can assume without loss of generality that $\eta_i = 1$ for $i \in F \cup \{1,2,3\}$. First we show $\|x\|_T = 1$. Well, since $\|x\|_T \geqslant \|x\|_0 = 1$, we just need to show $\|x\|_T \leqslant 1$.

By (8), if $\|x\|_T \neq \|x\|_0$, then $\|x\|_T = \sup\left\{\frac{1}{2}\sum\limits_{i=1}^{k} \|E_i x\|_T : k \in \mathbb{N}, k \geqslant 3, (E_i)_{i=1}^k \text{ admissible}\right\}$. So, for some $(E_i)_{i=1}^k$ admissible for $k \geqslant 3$, we have $\|x\|_T = \frac{1}{2}\sum_{i=1}^k \|E_i x\|_T$. Note that if $k \geqslant 4$, we are essentially working with a vector of all $\frac{1}{3}$'s with max supp $x = 12$ and Lemma 29 gives the norm of x as $\frac{1}{3}(\frac{6}{2})$, or 1, so we assume $k = 3$. Thus, $\|x\|_T = \frac{1}{2}\left(\|E_1 x\|_T + \|E_2 x\|_T + \|E_3 x\|_T\right)$.

By Lemma 19, we know each $\|E_i x\|_T \leqslant \|E_i x\|_0 \vee \frac{1}{2}\sum_{j \in E_i} |x(j)|$, so we have two options for each $\|E_i x\|_T$.
Case 1: Suppose $\|E_i x\|_T \leqslant \frac{1}{2}\sum_{j \in E_i} |x(j)|$ for all $i \in \{1,2,3\}$. Then

$$\|x\|_T = \frac{1}{2}\left(\|E_1 x\|_T + \|E_2 x\|_T + \|E_3 x\|_T\right) \leqslant \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2} + \frac{9}{3}\right)\right) = \frac{7}{8} < 1.$$

Case 2: Suppose $\|E_i x\|_T \leqslant \frac{1}{2}\sum_{j \in E_i} |x(j)|$ for two $i \in \{1,2,3\}$.

Case 2(a): Suppose $\|E_1 x\|_T \leqslant \|E_1 x\|_0 = \frac{1}{2}$. Then $\|E_2 x\|_T + \|E_3 x\|_T \leqslant \frac{1}{2}(\frac{9}{3})$. So,

$$\|x\|_T = \frac{1}{2}\left(\|E_1 x\|_T + \|E_2 x\|_T + \|E_3 x\|_T\right) \leqslant \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\left(\frac{9}{3}\right)\right) = \frac{1}{4} + \frac{1}{4}(3) = 1.$$

Case 2(b): Suppose $\|E_2 x\|_T \leqslant \|E_2 x\|_0 = \frac{1}{3}$. Then $\|E_1 x\|_T + \|E_3 x\|_T \leqslant \frac{1}{2}(\frac{1}{2} + \frac{8}{3})$. So,

$$\|x\|_T = \frac{1}{2}\left(\|E_1 x\|_T + \|E_2 x\|_T + \|E_3 x\|_T\right) \leqslant \frac{1}{2}\left(\frac{1}{3} + \frac{1}{2}\left(\frac{1}{2} + \frac{8}{3}\right)\right) = \frac{1}{6} + \frac{1}{4}\left(\frac{19}{6}\right) = \frac{23}{24} < 1.$$

Case 2(c): Suppose $\|E_3x\|_T \leqslant \|E_3x\|_0 = \frac{1}{3}$. The same argument used in Case 2(b) works to show $\|x\|_T \leqslant 1$ in this case.
Case 3: Suppose $\|E_ix\|_T \leqslant \frac{1}{2}\sum_{j\in E_i}|x(j)|$ for only one $i \in \{1,2,3\}$.

Case 3(a): Suppose $\|E_1x\|_T \leqslant \frac{1}{2}\sum_{j\in E_1}|x(j)|$. To get an upper bound, we suppose E_1x is a vector of $\frac{1}{2}$'s with $\max\operatorname{supp} E_1x = 10$ to get $\|E_1x\|_T \leqslant \frac{1}{2}(\frac{5}{2}) = \frac{5}{4}$. This upper bound holds by Lemma 29. Also, $\|E_2x\|_T = \|E_3x\|_T \leqslant \frac{1}{3}$, so

$$\|x\|_T = \frac{1}{2}\Big(\|E_1x\|_T + \|E_2x\|_T + \|E_3x\|_T\Big) \leqslant \frac{1}{2}\left(\frac{5}{4}+\frac{2}{3}\right) = \frac{5}{8}+\frac{1}{3} = \frac{23}{24} < 1.$$

Case 3(b): Suppose $\|E_2x\|_T \leqslant \frac{1}{2}\sum_{j\in E_2}|x(j)|$. By Lemma 29, $\|E_2x\|_T \leqslant \frac{1}{3}(\frac{6}{2}) = 1$. Also, $\|E_1x\|_T \leqslant \frac{1}{2}$ and $\|E_3x\|_T \leqslant \frac{1}{3}$, so

$$\|x\|_T = \frac{1}{2}\Big(\|E_1x\|_T + \|E_2x\|_T + \|E_3x\|_T\Big) \leqslant \frac{1}{2}\left(\frac{1}{2}+1+\frac{1}{3}\right) = \frac{1}{2}\left(\frac{11}{6}\right) = \frac{11}{12} < 1.$$

Case 3(c): Suppose $\|E_3x\|_T \leqslant \frac{1}{2}\sum_{j\in E_3}|x(j)|$. The same argument used in Case 3(b) works to show $\|x\|_T \leqslant 1$ in this case.
Case 4: Suppose $\|E_ix\|_T \leqslant \|E_ix\|_0$ for all $i \in \{1,2,3\}$. Then

$$\|x\|_T = \frac{1}{2}\Big(\|E_1x\|_T + \|E_2x\|_T + \|E_3x\|_T\Big) \leqslant \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\right) = \frac{1}{2}\left(\frac{7}{6}\right) = \frac{7}{12} < 1.$$

Therefore, $1 \leqslant \|x\|_T \leqslant 1$ in all cases, so $\|x\|_T = 1$.

Now suppose $x = \frac{1}{2}(y+z)$ for some $y, z \in B_T$. We will prove that $x = y = z$. Arguing as in the proof of Theorem 22 (Item 1), we know that $x(1) = x(2) = y(1) = y(2) = z(1) = z(2) = 1$. By Lemma 31, we know $x(k) = y(k) = z(k)$ for all $k \geqslant 6$. Now we show $x(5) = y(5) = z(5)$. We know $y(5) = \frac{1}{3} + \varepsilon_5\delta_5$ and $z(5) = \frac{1}{3} - \varepsilon_5\delta_5$ for some $\varepsilon_5 \in \{\pm 1\}$ and some $\delta_5 \geqslant 0$. Suppose via contradiction $\delta_5 > 0$. Then, supposing without loss of generality that $y(5) = \frac{1}{3} + \delta_5$, we see that the admissible sequence $(\{5\},\{6\},\{7\},\{8\},\{9,10,11,12\})$ yields

$$\begin{aligned}\|y\|_2 &\geqslant \frac{1}{2}\left(\left(\frac{1}{3}+\delta_5\right)+\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)+\left(\frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)\right)\right)\\ &= \frac{1}{2}(2+\delta_5) > 1.\end{aligned}$$

This contradicts the fact that $y \in B_T$. Therefore, $\delta_5 = 0$, as desired.

Now we want to show $y(4) = z(4) = \frac{1}{3}$. We know $y(4) = \frac{1}{3} + \varepsilon_4\delta_4$ and $z(4) = \frac{1}{3} - \varepsilon_4\delta_4$ for some $\varepsilon_4 \in \{\pm 1\}$ and some $\delta_4 \geqslant 0$. Suppose via contradiction $\delta_4 > 0$. Then, supposing without loss of generality that $y(4) = \frac{1}{3} + \delta_4$, we see that the admissible sequence $(\{4\},\{5\},\{6\},\{7,8,9,10,11,12\})$ yields

$$\|y\|_2 \geqslant \frac{1}{2}\left(\left(\frac{1}{3}+\delta_4\right)+\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)+\left(\frac{1}{2}\left(6\left(\frac{1}{3}\right)\right)\right)\right) = \frac{1}{2}(2+\delta_4) > 1.$$

This contradicts the fact that $y \in B_T$. Therefore, $\delta_4 = 0$, as desired. Now we want to show $y(3) = z(3) = \frac{1}{2}$. We know $y(3) = \frac{1}{2} + \varepsilon_3\delta_3$ and $z(3) = \frac{1}{2} - \varepsilon_3\delta_3$ for some $\varepsilon_3 \in \{\pm 1\}$ and some $\delta_3 \geqslant 0$. Suppose $\delta_3 > 0$ and suppose without loss of generality that $y(3) = \frac{1}{2} + \delta_3$. Then the admissible sequence $(\{3\}, \{4, 5, 6\}, \{7, 8, 9, 10, 11, 12\})$ yields

$$\|y\|_2 \geqslant \frac{1}{2}\left(\left(\frac{1}{2} + \delta_3\right) + \frac{1}{2}\left(3\left(\frac{1}{3}\right)\right) + \frac{1}{2}\left(6\left(\frac{1}{3}\right)\right)\right) = \frac{1}{2}\left(2 + \delta_3\right) > 1,$$

which contradicts $y \in B_T$. Thus, it must be that $y(3) = z(3) = \frac{1}{2}$. Putting it all together we have $x = y = z$, as desired. Therefore x is an extreme point of B_T by Definition 20. $\square$

7

IMPROVING BOUNDS ON $j(n)$

In this chapter we improve existing bounds on a quantity $j(n)$ introduced in [3].

Definition 32. *For n a positive integer, $j(n)$ is the smallest non-negative integer such that for all $x \in c_{00}$ with $\max \operatorname{supp} x \leqslant n$ we have*

$$\|x\|_{j(n)} = \max_{m \in \mathbb{N}} \|x\|_m$$

In [3] they state that $j(n) \leqslant \lfloor (n+1)/2 \rfloor$, admitting this is likely not a sharp upper bound for $j(n)$. In this chapter, we build on this idea to provide the following upper and lower bounds on $j(n)$.

Theorem 33. *The following items provide improved bounds on $j(n)$.*

1. *For all $n \in \mathbb{N}$, $j(n) > \log_2(n+1) - 4$.*
2. *For all $n \in \mathbb{N}$, $j(n) \leqslant 2\sqrt{n} + 5$.*

Before proving this theorem, we clarify what it means for a separate quantity $f(n)$ to be less than $j(n)$ or greater than or equal to $j(n)$. $f(n) < j(n)$ if there exists a $y \in c_{00}$ with $\max \operatorname{supp} y = n$ such that $\|y\|_{f(n)} < \|y\|_T$. On the contrary, $f(n) \geqslant j(n)$ if for all $y \in c_{00}$ with $\max \operatorname{supp} y = n$ we have $\|y\|_{f(n)} = \|y\|_T$.

We first prove the lower bound in the above theorem. From the definition of $j(n)$ we need to show that for each $n \in \mathbb{N}$ there is a vector $x_n \in c_{00}$ with $\max \operatorname{supp} x_n \leqslant n$ and $\|x_n\|_{\lfloor \log_2(n+1)-4 \rfloor} < \|x_n\|_T$. Such vectors exist within the cascade vectors, which come from the work of Noah Duncan [4]. We now introduce the following theorem to specify the cascade vectors we will need.

Theorem 34. *For each $n \in \mathbb{N}$ with $n \geqslant 3$ there exists a vector y_n such that $\max \operatorname{supp} y_n = 2^n - 1$ and there is an $f_n \in W_{n-2}$ such that $f_n(y_n) = \|y_n\|_T$ and for all $g \in W \setminus \{f_n\}$ we have $g(y_n) < \|y_n\|$.*

The above theorem follows from work in Duncan's thesis, so we will just present the definition of the y_n's. Letting $n \in \mathbb{N}$ with $n \geqslant 3$, we will define $y_n = c(n-2, 4)$, where $c(n-2, 4)$ is notation from Duncan's thesis. Instead of introducing this notation, we just define the first few y_n's.

$$y_3 = (0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots) = \frac{1}{2}\sum_{i=4}^{7} e_i,$$

$$\begin{aligned} y_4 &= (0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, 0, \ldots) \\ &= \frac{1}{2}\sum_{i=4}^{6} e_i + \frac{1}{8}\sum_{i=8}^{15} e_i, \end{aligned}$$

$$y_5 = \frac{1}{2}\sum_{i=4}^{6} e_i + \frac{1}{8}\sum_{i=8}^{14} e_i + \frac{1}{32}\sum_{i=16}^{31} e_i,$$

and so on. To help clarify what is actually happening as we go from one y_n to y_{n+1}, note that we take off y_n's last non-zero value and replace it with ($\max \operatorname{supp} y_n + 1$)-many values that sum to give a next level norm of that single replaced value, starting at index $\max \operatorname{supp} y_n + 1$, as a way to keep the norm constant. By design, $\max \operatorname{supp} y_n = 2^n - 1$.

The next proposition follows easily from the above theorem and will be useful in proving Theorem 33 (Item 1).

Proposition 35. *For each $n \in \mathbb{N}$ with $n \geqslant 3$, $j(2^n - 1) > n - 3$.*

Proof. Let $n \in \mathbb{N}$ with $n \geqslant 3$. We show that $j(2^n - 1) > n - 3$. Equivalently, we show that there exists some vector $x \in c_{00}$ with $\max \operatorname{supp} x = 2^n - 1$ and $\|x\|_{n-3} < \|x\|_T$. By Theorem 34, we know that the vector y_n has $\|y_n\|_{n-3} < \|y_n\|_{n-2} = \|y_n\|_T$, so y_n fits this desired description. □

With the above proposition in place, our proof of Theorem 33 (Item 1) does not require much more work, as seen below.

Proof. (Theorem 33)(Item 1) Let $k \in \mathbb{N}$. If $k \leqslant 7$, we are trivially done, since $\log_2(k+1) - 4 < 0$. Thus we assume $k > 7$. Note that we have $2^{n-1} - 1 \leqslant k < 2^n - 1$ for some $n \geqslant 4$. Equivalently, $2^{n-1} \leqslant k + 1 < 2^n$. So $n - 1 \leqslant \log_2(k+1) < n$. Also note that for all $m \in \mathbb{N}$, $j(m) \leqslant j(m+1)$. In other words, $j(m)$ increases as m increases. Therefore,

$$j(k) \geqslant j(2^{n-1} - 1) > (n-1) - 3 = n - 4 > \log_2(k+1) - 4.$$

The first inequality follows from the facts that $k \geqslant 2^{n-1} - 1$ and $j(m)$ increases as m increases. The second inequality comes from Proposition 35. The last inequality comes from the fact that $\log_2(k + 1) < n$. Therefore, $j(k) > \log_2(k+1) - 4$, as desired. □

We now consider upper bounds on $j(n)$ with the goal of proving Theorem 33 (Item 2). We introduce the following notation.

Denote $\mathbb{N}^{<\mathbb{N}} = \cup_{n=1}^{\infty} \mathbb{N}^n$. For $\sigma \in \mathbb{N}^{<\mathbb{N}}$, if $\sigma = (\sigma(1), \ldots, \sigma(k))$, we have $|\sigma| = k$ and $s(\sigma) := \sigma(1) + \cdots + \sigma(k) = \sum_{i=1}^{k} \sigma(i)$. We will also say $|\emptyset| = 0$.

Definition 36 (Tree Index Set)**.** *For each $f \in W$ there is a set $\mathcal{T}_f \subseteq \mathbb{N}^{<\mathbb{N}} \cup \{\emptyset\}$ called the tree index set of f. We set $f = f_\emptyset$.*

1. *$\sigma \in \mathcal{T}_f$ is called a terminal node if $\sigma \frown 1 \notin \mathcal{T}_f$. If σ is a terminal node then $f_\sigma = \pm e_i^*$ for some $i \in \mathbb{N}$.*
2. *If $\sigma \in \mathcal{T}_f$ is not a terminal node, then*

$$f_\sigma = \frac{1}{2} \sum_{\{k:\sigma\frown k \in \mathcal{T}_f\}} f_{\sigma \frown k},$$

where $\{k : \sigma \frown k \in \mathcal{T}_f\} = \{1, \ldots, m\}$ for some $m \in \mathbb{N}$.

As an example to get familiar with the above notation, if $f = \frac{1}{2}(\frac{1}{2}(e_3^* + e_4^* + e_5^*) + e_7^* + e_8^*)$, we say $f_{(1)} = \frac{1}{2}(e_3^* + e_4^* + e_5^*)$, $f_{(1,1)} = e_3^*$, $f_{(1,2)} = e_4^*$, $f_{(1,3)} = e_5^*$, $f_{(2)} = e_7^*$, and $f_{(3)} = e_8^*$. So, $\mathcal{T}_f = \{\emptyset, (1), (2), (3), (1,1), (1,2), (1,3)\}$.

With this notation formalized, we introduce the following remark.

Remark 37. For $f \in W$, let $m = \max\{|\sigma| : \sigma \in \mathcal{T}_f\}$. Then $f \in W_m$.

We introduce the below lemma, whose proof we will postpone until later.

Lemma 38. *For $f \in W$ and $\sigma \in \mathcal{T}_f$, let $\ell = \min \operatorname{supp} f$ and $n \geqslant \max \operatorname{supp} f$. Then*

$$|\text{supp } f_\sigma| - \min \operatorname{supp} f_\sigma \leqslant n - (|\sigma| + 2)(\ell - 1) - \left(\sum_{i=1}^{|\sigma|} [|\sigma| + 1 - i][\sigma(i) - 1] \right) - 1.$$

The above lemma will be used to prove the following proposition, which provides a similar upper bound on $j(n)$ to the one found in [3]. We include the proof of this proposition here because it helped set up the methodology to find an even better upper bound.

Proposition 39. *For $x \in c_{00}$ with $\max \operatorname{supp} x = n$ and $f \in W$ such that $f(x) = \|x\|_T$, $f \in W_{\lceil \frac{n}{2} \rceil}$.*

Proof. Let $x \in c_{00}$ with $\max \operatorname{supp} x = n$ and $f \in W$ such that $f(x) = \|x\|_T$. We begin by showing that $|\sigma| \geqslant \frac{n}{2} - 2$ implies $f_\sigma \in W_1$. Suppose $|\sigma| \geqslant \frac{n}{2} - 2$. By Lemma 38,

$$|\text{supp } f_\sigma| - \min \operatorname{supp} f_\sigma \leqslant n - (|\sigma| + 2)(\ell - 1) - \left(\sum_{i=1}^{|\sigma|} [|\sigma| + 1 - i][\sigma(i) - 1] \right) - 1,$$

where $\ell = \min \operatorname{supp} f$. Note that $(|\sigma| + 2)(\ell - 1) \geqslant (n/2)(2) = n$, using our initial assumption about $|\sigma|$ and the fact that $\ell \geqslant 3$ by (8). Also $\sigma(i) - 1 = 0$ is a possibility but $|\sigma| + 1 - i \geqslant 0$, so we have

$$|\operatorname{supp} f_\sigma| - \min \operatorname{supp} f_\sigma \leqslant n - n - 0 - 0 = 0.$$

Then $f_\sigma \in W_1$ since $|\operatorname{supp} f_\sigma| - \min \operatorname{supp} f_\sigma \leqslant 0$ implies this fact by definition of W_1.

Assume via contradiction $f \notin W_{\lceil \frac{n}{2} \rceil}$. Now suppose $m = \max\{|\sigma| : \sigma \in \mathcal{T}_f\} > \lceil \frac{n}{2} \rceil$. Find $\sigma = (n_1, \dots, n_k)$ such that $k > \lceil \frac{n}{2} \rceil$. Then $k - 2 > \frac{n}{2} - 2$, which implies $f_{(n_1,\dots,n_{k-2})} \in W_1$, as explained earlier. Therefore, it cannot be that $(n_1, \dots, n_k) \in \mathcal{T}_f$, which contradicts the existence of a $\sigma \in \mathcal{T}_f$ such that $|\sigma| = k$. Thus, $f \in W_{\lceil \frac{n}{2} \rceil}$, as desired. □

With the above lemma and proposition in place, we now prove a theorem to allow us to move over functionals' minimimum supports as we go deeper into their breaking. To do so, we begin by defining the following term for a specific type of functional.

Definition 40. *Let $x \in c_{00}$. $f \in W$ is a BH-functional for x if:*

1. $f(x) = \|x\|_T$
2. $\|x|_{[m+1,\infty)}\|_T < f(x)$ *for* $m = \min \operatorname{supp} f$.

Theorem 41. *Let $x \in c_{00}$ and suppose there is an $f \in W$ with $(1,1) \in \mathcal{T}_f$ such that f is a BH-functional for x. Then $f_{(1,1)} \in W_1$.*

Proof. Let $x \in c_{00}(\mathbb{N})$ and find $f \in W$ such that f is a BH-functional for x, where $m = \min \operatorname{supp} f$. Assume that $(1,1) \in \mathcal{T}_f$ and that $f_{(1,1)} \notin W_1$. Then $\min \operatorname{supp} f_{(1,2)} \geqslant 2m$. Note we assume without loss of generality that $(1,2) \in \mathcal{T}_f$. Let

$$g = \frac{1}{2}(f_{(1,2)} + \cdots + f_{(1,m)} + f_{(2)} + \cdots + f_{(m)}),$$

$$h = \frac{1}{2}(f_{(1,1)} + f_{(2)} + \cdots f_{(m)}).$$

Note that $g, h \in W$. By definition $f_{(1)}(x) = \frac{1}{2}(f_{(1,1)}(x)) + \frac{1}{2}(f_{(1,2)} + \cdots + f_{(1,m)})(x)$. By assumption, $g(x) \leqslant \|x|_{[m+1,\infty)}\|_T < f(x)$. Thus $(f_{(1,2)} + \cdots + f_{(1,m)})(x) < f_{(1)}(x)$, which implies $f_{(1,1)}(x) > f_{(1)}(x)$. However, this yields $h(x) > f(x)$, which cannot happen since $h \in W$ and $f(x) = \|x\|_T$. This is the desired contradiction. □

Before proving Theorem 33 (Item 2), we introduce the following lemma.

Lemma 42. *Let $x \in c_{00}$. Then we can find $f \in W$ such that for all $\sigma \in \mathcal{T}_f$ we have f_σ as a BH-functional for $E_\sigma x$ where $E_\sigma = \operatorname{supp} f_\sigma$.*

We will not prove this lemma, but it follows from induction on the height of the tree. We are now ready to prove the main theorem of this chapter.

Proof. (Theorem 33)(Item 2) Recall that our goal is to show $j(n) \leqslant 2\sqrt{n}+5$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. If $n \leqslant 3$, we know from (8) that the zero norm will norm all vectors with maximum support of n, so we assume $n \geqslant 4$. Let $x \in c_{00}$ with $\max \operatorname{supp} x = n$. Find $f \in W$ such that for all $\sigma \in \mathcal{T}_f$ we have f_σ as a BH-functional for $E_\sigma x$. We know we can do this by Lemma 42. Then for $m = \max\{|\sigma| : \sigma \in \mathcal{T}_f\}$ we have $f \in W_m$ by Remark 37. Let $\ell = \min \operatorname{supp} f$ and $n \geqslant \max \operatorname{supp} f$.

By Lemma 38, we know that for all $\sigma \in \mathcal{T}_n$

$$|\operatorname{supp} f_\sigma| - \min \operatorname{supp} f_\sigma \leqslant n - (|\sigma|+2)(\ell-1) - \left(\sum_{i=1}^{|\sigma|}[|\sigma|+1-i][\sigma(i)-1]\right) - 1.$$

Now we would like to get a good lower bound for the term $\sum_{i=1}^{|\sigma|}[|\sigma|+1-i][\sigma(i)-1]$. To do so we need to exclude the possibility of $\sigma(i)=1$ for all i or $\sigma = (1,\ldots,1)$. We will show that for all $\sigma \in \mathcal{T}_f$

$$\left(\sum_{i=1}^{|\sigma|}[|\sigma|+1-i][\sigma(i)-1]\right) \geqslant \left(\frac{|\sigma|-3}{2}\right)^2,$$

which will give us the following:
For $|\sigma| \geqslant 2\sqrt{n}+3$,

$$\begin{aligned}
|\operatorname{supp} f_\sigma| - \min \operatorname{supp} f_\sigma &\leqslant n - (|\sigma|+2)(\ell-1) - \left(\sum_{i=1}^{|\sigma|}[|\sigma|+1-i][\sigma(i)-1]\right) - 1 \\
&\leqslant n - 0 - \left(\frac{|\sigma|-3}{2}\right)^2 - 0 \\
&\leqslant n - \left(\frac{2\sqrt{n}+3-3}{2}\right)^2 \\
&\leqslant 0.
\end{aligned}$$

Thus $f_\sigma \in W_1$.

Then since $|\sigma| \geqslant 2\sqrt{n}+3$ implies $f_\sigma \in W_1$, we can prove that $f \in W_{\lceil 2\sqrt{n}+5 \rceil}$. Assume via contradiction $f \notin W_{\lceil 2\sqrt{n}+5 \rceil}$. Now suppose $m = \max\{|\sigma| : \sigma \in \mathcal{T}_f\} > 2\sqrt{n}+5$. Find $\sigma = (n_1,\ldots,n_k)$ such that $k > 2\sqrt{n}+5$. Then $k-2 > 2\sqrt{n}+5-2$, which implies $f_{(n_1,\ldots,n_{k-2})} \in W_1$, as explained earlier. Therefore, it cannot be that $f_{(n_1,\ldots,n_k)} \in \mathcal{T}_f$, which contradicts the existence of a $\sigma \in \mathcal{T}_f$ such that $|\sigma| = k$. Thus, $f \in W_{\lceil 2\sqrt{n}+5 \rceil}$, as desired.

It remains to prove that for all $\sigma \in \mathcal{T}_f$

$$\left(\sum_{i=1}^{|\sigma|}[|\sigma|+1-i][\sigma(i)-1]\right) \geqslant \left(\frac{|\sigma|-3}{2}\right)^2,$$

To do so, we will need the following claim.

Claim 43. *If $\sigma \in \mathcal{T}_f$ and there is a $j \in \mathbb{N}$ such that $\sigma(j) = \sigma(j-1) = 1$, then $|\sigma| \leqslant j+1$.*

As a result of the above claim, if a pair of consecutive 1's exists within a σ, the 1's have to be in the third-to-last and second-to-last positions within σ and/or in σ's last two positions. Denote the set of all σ's in $\mathbb{N}^{<\mathbb{N}} \cup \{\emptyset\}$ that do not contradict the above condition as $\mathcal{T}_{big}$. Then

$$\begin{aligned}
\min\left\{\sum_{i=1}^{|\sigma|}[|\sigma|+1-i][\sigma(i)-1] : \sigma \in \mathcal{T}_f\right\} &\geqslant \min\left\{\sum_{i=1}^{|\sigma|}[|\sigma|+1-i][\sigma(i)-1] : \sigma \in \mathcal{T}_{big}\right\} \\
&\geqslant [|\sigma|][0] + [|\sigma|-1][1] + [|\sigma|-2][0] \\
&\qquad + [|\sigma|-3][1] + \cdots + [4][1] + [3][0] + [2][0] + [1][0] \\
&= 2+4+6+8+\cdots+(|\sigma|-1)-2 \\
&= 2\left(1+2+3+4+\cdots+\frac{|\sigma|-1}{2}\right)-2 \\
&= 2\left(\frac{\left(\frac{|\sigma|-1}{2}\right)\left(\frac{|\sigma|-1+2}{2}\right)}{2}\right)-2 \\
&= \frac{(|\sigma|-1)(|\sigma|+1)}{4}-2 \\
&= \frac{|\sigma|^2-9}{4} \\
&\geqslant \left(\frac{|\sigma|-3}{2}\right)^2.
\end{aligned}$$

The first inequality above follows from the facts that $\mathcal{T}_f \subseteq \mathcal{T}_{big}$ and taking a minimum over a smaller set of σ's will result in a minimum at least as big as the minimum over the larger set. The second relation above results from considering the $\sigma \in \mathcal{T}_{big}$ yielding the smallest possible summation of $[|\sigma|+1-i][\sigma(i)-1]$'s, which occurs when we have as many 1's as possible in σ and the rest as 2's. In particular, this $\sigma = (1,2,1,2,\ldots,1,1,1)$. By assuming $|\sigma|$ is odd, we ensure the fourth-to-last coordinate in our σ and our summation is non-zero. This estimate is good enough, since adding in another coordinate to make $|\sigma|$ even would just result in a non-zero fifth-to-last coordinate and a zero fourth-to-last coordinate.

The third relation above follows from adding and subtracting 2 to our summation, and the fourth relation simplifies our expression. The fifth relation relies on the fact that summing the first n natural numbers yields $\frac{n(n+1)}{2}$. Note that we can count on $\frac{|\sigma|-1}{2}$ being a natural number here, since $|\sigma|$ is assumed to be odd. The next two lines involve simplifying expressions, while the last relation uses the fact that $|\sigma|^2-9 = (|\sigma|-3)(|\sigma|+3) \geqslant (|\sigma|-3)(|\sigma|-3)$. This proves that for all $\sigma \in \mathcal{T}_f$

$$\left(\sum_{i=1}^{|\sigma|}[|\sigma|+1-i][\sigma(i)-1]\right) \geqslant \left(\frac{|\sigma|-3}{2}\right)^2,$$

which finishes the proof. $\square$

Now we finally get around to proving Lemma 38, which was essential to proving the above main theorem for this chapter.

Proof. (Lemma 38) Recall that our goal is to prove

$$|\text{supp } f_\sigma| - \min \text{supp } f_\sigma \leqslant n - (|\sigma|+2)(\ell-1) - \left(\sum_{i=1}^{|\sigma|} [|\sigma|+1-i][\sigma(i)-1] \right) - 1,$$

where $\ell = \min \text{supp } f$ and $n \geqslant \max \text{supp } f$. To do so, we will need the following inequalities, which we will prove hold.

$$\min \text{supp } f_\sigma \geqslant \ell + s(\sigma) - |\sigma| \tag{10}$$

$$|\text{supp } f_\sigma| \leqslant n - (|\sigma|+1)(\ell-1) - \sum_{i=1}^{|\sigma|-1} [|\sigma|-i][\sigma(i)-1], \tag{11}$$

To prove (10), we let $|\sigma| = k$ and use induction on k. Let $\sigma|_{k-1} = (n_1, \ldots, n_{k-1})$ if $\sigma = (n_1, \ldots, n_{k-1}, n_k)$. In the base case of $|\sigma| = 1$, we know $\min \text{supp } f_\sigma \geqslant \ell + s(\sigma) - 1$, since there are at least $(s(\sigma)-1)$-many values from ℓ to f_σ's beginning index (assuming all prior functionals g_σ with the same $|\sigma|$ are all e_i^*'s). Now we assume $\min \text{supp } f_\sigma \geqslant \ell + s(\sigma) - |\sigma|$ for some $|\sigma| = k \in \mathbb{N}$ and show the same inequality holds for $|\sigma| = k+1$.

$$\begin{aligned}
\min \text{supp } f_\sigma &\geqslant \min \text{supp } f_{\sigma|_k} + \sigma(k+1) - 1 \\
&\geqslant \ell + s(\sigma|_k) - |\sigma|_k| + \sigma(k+1) - 1 \\
&= \ell + s(\sigma) - (k+1).
\end{aligned}$$

The first inequality relies on the fact that f_σ can have the same minimum support value as $f_{\sigma|_k}$ if $\sigma(k+1) = 1$. The second inequality above follows from the inductive hypothesis. The lone equality above follows from the facts that $s(\sigma|_k) + \sigma(k+1) = s(\sigma)$ and $|\sigma|_k| = k$. Thus, (10) holds.

To prove (11), we let $|\sigma| = k$ and use induction on k. Let $\sigma|_{k-1} = (n_1, \ldots, n_{k-1})$ if $\sigma = (n_1, \ldots, n_{k-1}, n_k)$. It follows that

$$\begin{aligned}
|\text{supp } f_\sigma| &\leqslant |\text{supp } f_{\sigma|_{k-1}}| - (\min\text{supp } f_{\sigma|_{k-1}}) + 1 \\
&\leqslant n - k(\ell-1) - \left(\sum_{i=1}^{k-2}[(k-1)-i][\sigma(i)-1]\right) - \ell - s(\sigma|_{k-1}) + (k-1) + 1 \\
&\leqslant n - (k+1)(\ell-1) - \left(\sum_{i=1}^{k-2}[(k-1)-i][\sigma(i)-1]\right) - \sum_{i=1}^{k-1}\sigma(i) + \sum_{i=1}^{k-1} 1 \\
&\leqslant n - (k+1)(\ell-1) - \left(\sum_{i=1}^{k-2}[(k-1)-i][\sigma(i)-1]\right) - \sum_{i=1}^{k-1}[\sigma(i)-1] \\
&\leqslant n - (k+1)(\ell-1) - \left(\sum_{i=1}^{k-2}\Big[[(k-1)-i][\sigma(i)-1] + [\sigma(i)-1]\Big]\right) \\
&\qquad - [\sigma(k-1)-1] \\
&= n - (k+1)(\ell-1) - \left(\sum_{i=1}^{k-2}[k-i][\sigma(i)-1]\right) - [\sigma(k-1)-1][k-(k-1)] \\
&= n - (k+1)(\ell-1) - \sum_{i=1}^{k-1}[k-i][\sigma(i)-1].
\end{aligned}$$

The first line above is clearly true. The second line relies on our inductive hypothesis for $k-1$ in combination with (10). The third line follows from our definition of the sum function s. The remaining lines above follow from simplifying expressions.

Now,

$$\begin{aligned}
|\text{supp } f_\sigma| - \min\text{supp } f_\sigma &\leqslant n - (k+1)(\ell-1) - \left(\sum_{i=1}^{k-1}[k-i][\sigma(i)-1]\right) \\
&\qquad - \ell - s(\sigma) + |\sigma| \\
&= n - (k+1)(\ell-1) - \left(\sum_{i=1}^{k-1}[k-i][\sigma(i)-1]\right) - (\ell-1) \\
&\qquad - \left(\sum_{i=1}^{k}\sigma(i)\right) + k - 1 \\
&= n - (k+1)(\ell-1) - \left(\sum_{i=1}^{k-1}[k-i][\sigma(i)-1]\right) - (\ell-1) \\
&\qquad - \left(\sum_{i=1}^{k}[\sigma(i)-1]\right) - 1 \\
&= n - (k+2)(\ell-1) - \left(\sum_{i=1}^{k-1}[(k-i)+1][\sigma(i)-1]\right) - [\sigma(k)-1] - 1 \\
&= n - (k+2)(\ell-1) - \left(\sum_{i=1}^{k}[(k+1)-i][\sigma(i)-1]\right) - 1.
\end{aligned}$$

The first line above comes from applying (10) and (11). The second line involves adding and subtracting 1 in addition to simplifying expressions, and the rest of the above relations result from algebra. $\square$

8

HELPFUL CODE

While working on this thesis, we had to perform many difficult computations involving norms. To expedite the computational process, we created the below Python files, whose purposes are laid out. Click on the links to see the code.

1. **norm_plus_breakings.py**
   Given a vector and a norm level n, this program finds the vector's n-level norm value and the breakings that yield this value.
   `https://github.com/holtm16/HonorsThesis/blob/master/norm_plus_breakings.py`

2. **extreme_pts.py**
   Given a vector and a norm level n, this program finds the vectors that differ from the input vector by one or more slightly-altered coordinates and maintain the same n-level norm value.
   `https://github.com/holtm16/HonorsThesis/blob/master/extreme_pts.py`

3. **just_norm_value.py**
   Given a vector and a norm level n, this program simply finds the vector's n-level norm value.
   `https://github.com/holtm16/HonorsThesis/blob/master/just_norm_value.py`

4. **breaking_possibilities.py**
   Given a vector length n with $n \geqslant 7$, this program finds all possible ways to break a vector of length n.
   `https://github.com/holtm16/HonorsThesis/blob/master/breaking_possibilities.py`

For an explanation of how to run the above Python programs, download the Microsoft Word document from the below URL (click "View Raw" on the page to begin the download).
`https://github.com/holtm16/HonorsThesis/blob/master/README.docx`

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