

To deduce the equation of the tangent plane to the surface $u = f(x, y, z)$ at the point $P(x_1, y_1, z_1)$

The general equation of a plane is $Ax + By + Cz + D = 0$

If it passes through the point $P(x_1, y_1, z_1)$ we will have

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \quad (1)$$

We may determine the values of A, B, C by imposing conditions of tangency. Suppose (1) to be tangent at P .

Let the surface and tangent plane be intersected by planes passing through P , and parallel respectively to xz and yz .

The equations of the line cut from the tangent plane by the plane parallel to xz will be

$$x - x_1 = t(z - z_1) \quad (2) \text{ and } y = y_1 \quad (3)$$

and of the line cut by the plane parallel to yz

$$y - y_1 = s(z - z_1) \quad (4) \text{ and } x = x_1 \quad (5)$$

The tangent plane will contain these lines.

The trace of (1) on xz is

$$A(x - x_1) - B(y_1) + C(z - z_1) = 0 \quad (6)$$

$$\text{And on } yz, -Ax_1 + B(y - y_1) + C(z - z_1) = 0 \quad (7)$$

But (6) is parallel to (2)

and (7) - - - - (4)

$\therefore t = -\frac{C}{A}$ and $s = -\frac{C}{B}$, and (1) becomes

$$z - z_1 = \frac{1}{t}(x - x_1) + \frac{1}{s}(y - y_1) \quad (8)$$

Since (2), (3) and (4), (5) are respectively tangent to the corresponding curves cut from the surface, we have

$$t = \frac{\partial x_1}{\partial z_1} \text{ and } s = \frac{\partial y_1}{\partial z_1} \text{ or } \frac{1}{t} = \frac{\partial z_1}{\partial x_1} \text{ and } \frac{1}{s} = \frac{\partial z_1}{\partial y_1}$$

$$\text{And (8) becomes } z - z_1 = \frac{\partial z_1}{\partial x_1}(x - x_1) + \frac{\partial z_1}{\partial y_1}(y - y_1) \quad (9)$$

$\frac{dz_1}{dx_1}$ and $\frac{dz_1}{dy_1}$ are the partial differential coeffs derived from the equation of the surface, and they will have the same values at the point P as the similar coeffs derived from the equation of the tangent plane at that point.

Since the equation of the surface is $u = f(x, y, z) = 0$

$$\left[\frac{du}{dx} \right] = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} = 0, \text{ and } \left[\frac{\partial u}{dy} \right] = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} = 0$$

$$\text{and } \frac{\partial z_1}{\partial x_1} = - \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial z}}, \quad \frac{\partial z_1}{\partial y_1} = - \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial z}}$$

Substituting in (9) we have

$$(x - x_1) \frac{\partial u}{\partial x_1} + (y - y_1) \frac{\partial u}{\partial y_1} + (z - z_1) \frac{\partial u}{\partial z_1} = 0$$

as in Bayley.

A line from the origin perpendicular to the tangent plane will be, $x = tz, y = sz$

The projections of this line are perpendicular to the traces of the plane.

$$\therefore t = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial z_1}} \quad \& \quad s = \frac{\frac{\partial u}{\partial y_1}}{\frac{\partial u}{\partial z_1}}$$

Let $x' y' z'$ be a point on the perpendicular, and r' its distance from the origin.

$$\begin{aligned} x' &= tz' \\ y' &= sz' \\ r'^2 &= x'^2 + y'^2 + z'^2 \\ &= t^2 z'^2 + s^2 z'^2 + z'^2 \\ &= z'^2 (1 + t^2 + s^2) \\ \therefore z' &= \frac{r'}{\sqrt{1 + t^2 + s^2}} \end{aligned}$$

$$\begin{aligned} x' &= \frac{tr'}{\sqrt{1 + t^2 + s^2}} \\ y' &= \frac{sr'}{\sqrt{1 + t^2 + s^2}} \\ \text{But } \cos \alpha &= \frac{x'}{r'} = \frac{t}{\sqrt{1 + t^2 + s^2}} \end{aligned}$$

$$\cos \beta = \frac{s}{\sqrt{1 + t^2 + s^2}}$$

$$\cos \gamma = \frac{1}{\sqrt{1 + t^2 + s^2}}$$

$$\begin{aligned} \text{or } \cos \alpha &= \frac{\frac{\partial u}{\partial x_1}}{\sqrt{1 + \left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial z_1} \right)^2}} \\ &= \frac{\frac{\partial u}{\partial x_1}}{\sqrt{\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial y_1} \right)^2 + \left(\frac{\partial u}{\partial z_1} \right)^2}} \end{aligned}$$

and $\cos \beta$

$$\cos \beta = \frac{\frac{\partial u}{\partial y_1}}{\sqrt{\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial y_1} \right)^2 + \left(\frac{\partial u}{\partial z_1} \right)^2}}$$

$$\cos \gamma = \frac{\frac{\partial u}{\partial z_1}}{\sqrt{\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial y_1} \right)^2 + \left(\frac{\partial u}{\partial z_1} \right)^2}}$$