III [redacted]

III. $\int \frac{dy}{1-x^3} = \int \frac{dy}{1-x} + \int \frac{dy}{x^2+x+1}$

third & first class.

$\frac{1}{(1-x)(x^2+x+1)} = \frac{A}{1-x} + \frac{Bx+C}{x^2+x+1}$

$1 = Ax^2 + Ax + A + Bx + C - Bx^2 - Cx$

now arrange in descending powers

$1 = x^2(A-B) + x(A+B-C) + A + C$

$A - B = 0 \qquad A = \frac{1}{3}$

$A + B - C = 0 \qquad B = \frac{1}{3}$

$A + C = 1 \qquad C = \frac{2}{3}$

$\int \frac{dy}{1-x^3} = \frac{1}{3}\int \frac{dy}{1-x} + \frac{1}{3}\int \frac{\cancel{2}(x+2)\,dy}{x^2+x+1}.$

$= -\frac{1}{3}\int \frac{dy}{x-\cancel{x}} + \frac{1}{3}\int \frac{x\,dy}{x^2+x+1} + 2\int \frac{dx}{x^2+x+1}$

$= -\frac{1}{3}\log(x-1) + \frac{1}{3}\int \frac{x\,dy}{x^2+x+1} + 2\int \frac{dy}{x^2+x+1}$

10 $\qquad \frac{1}{3}\int \frac{x\,dx}{x^2+x+1} = ? \qquad 2\int \frac{dy}{x^2+x+1} = ?$

There remains the term $8\sqrt{2}\,a^2\int_0^{\frac{\pi}{2}}\frac{d\varphi}{1+\sin^2\varphi}$

$$\int_0^{\frac{\pi}{2}}\frac{d\varphi}{1+\sin^2\varphi} = \int_0^{\frac{\pi}{2}}\frac{d\varphi}{1+\frac{\tan^2\varphi}{\sec^2\varphi}} = \int_0^{\frac{\pi}{2}}\frac{\sec^2\varphi\,d\varphi}{\sec^2\varphi+\tan^2\varphi} = \int_0^{\frac{\pi}{2}}\frac{\sec^2\varphi\,d\varphi}{1+2\tan^2\varphi}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{\sqrt{2}\sec^2\varphi\,d\varphi}{1+2\tan^2\varphi} = \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}\tan\varphi = \frac{1}{\sqrt{2}}\frac{\pi}{2}$$

And $8\sqrt{2}\,a^2\int_0^{\frac{\pi}{2}}\frac{d\varphi}{1+\sin^2\varphi} = 4\pi a^2$

$\therefore$ The total area $= 4\pi a^2 - 5\sqrt{2}\,a^2\frac{\pi}{2}$ or

Area $= \pi a^2\left(4 - \frac{5\sqrt{2}}{2}\right)$