ON THE REALIZABILITY OF PROJECTIVE CONFIGURATIONS

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CHAPTER 1. INTRODUCTION

1.1. The Parable of Campestria

Nestled in the Quadratus Mountains, there was once a great kingdom that thrived for centuries before its collapse. Campestria, as the kingdom is known to us, attributed her prosperity almost entirely to geography. The Campestrian lands were completely flat and her border formed a perfect rectangle, bounded by a treacherous mountain range whose snow-capped peaks melted into gentle streams in the spring. Chilly waters snaked down the mountains' slopes and uniformly irrigated the Campestrian fields each year. Her communal harvests were consistent and bountiful, and in turn, her people were jovial and content, albeit uneducated.

For many years, Campestria flourished under the rule of Paritius the Sixth, the latest of a long and proud dynasty. The old man was well-liked – his people found him compassionate and strong-willed, though not once unfair – but the Campestrians grew anxious as his beard whitened, for Paritius lacked a successor. Each day at sunset, the king fell to his knees in prayer, begging the patron goddess of Campestria for an heir. "Let not my fathers' line end at Six" he pled. "Bring Seven to the Paritians!"

On the first day of spring, when the mountaintops began to melt, the king awoke to find seven identical boys placed in a crib at his doorestep. Paritius wept, for the goddess had answered his prayers: these infants were to be his sons. Ever in support of fairness, though, the king realized a greater task was now at hand: Campestria had to be divided into seven regions of equal area, one for each son. Paritius knew he could accomplish this goal with rectangular regions, but feared that in this case an overzealous governor could build walls along his borders and block water from reaching the neighboring regions. Wisely, then, Paritius set out to separate Campestria into seven triangular regions, each with equal area. "How hard could it be?" he thought. "All that's required is the placement of some points and some lines within a rectangular boundary." He spent the whole day attempting to separate his kingdom, but to no avail. Frustrated by his failure, Paritius mandated that every man in the kingdom submit to him a valid plan: the best drawing would allocate Campestria for his sons.

One week later, Paritius set out into his kingdom to collect the plans. He stopped first at a small cottage, where he was greeted by a modest farmer. The king entered into the cottage and immediately demanded a glass of wine, so that the two might drink together while admiring the farmer's drawing. The farmer scrambled to fetch the wine and timidly said "The task was impossible, sir." He had made no plan. Outraged, Paritius accused the man of treason and condemned him to exile over the mountains. The farmer begged Paritius to stay for the afternoon, to drink, and to reconsider his sentence, but the king replied "My decision is final. Our drink together will be your last in Campestria." The farmer became incensed (the punishment was unfair!) and slipped poison into the wine, knowing that his exile was a death sentence in all but name. Thus began the collapse of Campestria.

Following the king's death, his seven infant sons were elevated to the throne simultaneously, as no alternative arrangements had been made. As they aged, rivalry and jealousy broke out between them. Campestria was soon ripped apart by a series of bloody civil wars, and the once-glorious kingdom would never rise from the conflicts.

It would be easy to debate whether jealousy or greed was the primary cause of Campestria's demise, as indeed philsophers and historians are so often accustomed, but Paul Monsky proved both arguments to be unfounded in 1970 [12]. He showed that the farmer was right all along: Paritius had requested the impossible. One cannot divide a square (or rectangle, as will be discussed in Chapter 2.2) into an odd number of triangles with equal areas. Had more effort been used to understand the nuances of placing points and lines, the downfall of Campestria may have been avoided. Welcome, dear reader, to my thesis. It's a story about points and lines.

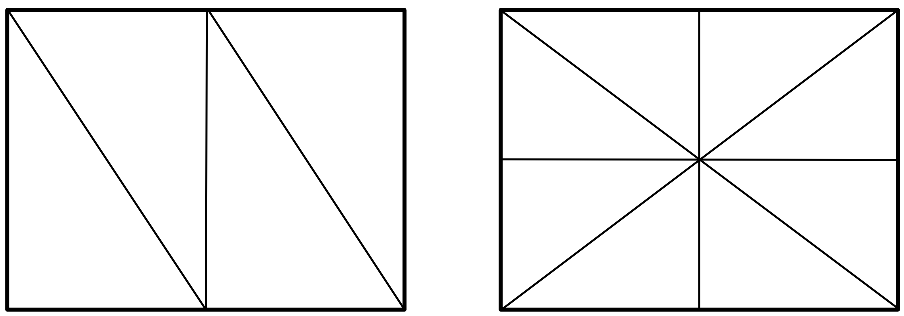

Figure 1. Equidissections of a rectangle (like Campestria) are created by placing points and lines within its boundary. Monsky's Theorem shows that there is no equidissection of a rectangle into an odd number of triangles [12].

1.2. Points and Lines and Incidence (oh my!)

When faced with complicated problems, scientists from many disciplines often reduce the "trickier" aspects into simpler, more manageable elements. Physicists learn the behavior of a system by understanding the effects of the many individual forces at play; ecologists comprehend ecosystems via the interrelationships of all the life they contain; and neuroscientists approach memory, behavior, and consciousness through the structures and functions of neurons. In a similar manner, many problems in ancient and modern geometry can be boiled down to the relationships between points and lines, subject to various constraints. For example, the parable in Section 1.1 describes a question originally posed to mathematicians as the *equidissection problem*: can a rectangle be dissected into an odd number of non-overlapping triangles, each with equal area? As Paritius described in the parable, this problem can be approached by placing points within the boundary of a rectangle and connecting them with line segments, subject to the constraints that the connected regions must be triangular, that there must be an odd number of these regions, and that each region has an equal area. Monsky's proof makes use of this approach, focusing on vertices and edges instead of triangles [12].

In other words, when faced with a problem in geometry pertaining to two-dimensional objects, it is often helpful to examine the *configuration* of their one-and zero-dimensional boundary components: points and lines. The equidissection problem and many others like it can therefore be re-framed as questions of whether or not a particular configuration (or family of configurations) is *realizable*. This thesis will focus almost exclusively on the realizability of various configurations, a topic which, like much of mathematics, has surprising depth and richness despite its easily-accessible appearance. To discuss realizability is to discuss incidence – that is, which points lie on which lines – and so this section will introduce the relationship between incidence theorems and geometric configurations.

First, we must introduce some basic concepts and notations. As noted repeatedly above, points and lines are the key players in our discussion of incidence theorems. It's not an easy task to properly define these objects, as they seem so pictorially intuitive. In his *Elements*, Euclid defines a point as "that which has no part." If that definition makes no sense to you, dear reader, you're not alone. Imagine instead a pinprick in a piece of paper that, no matter how greatly magnified, remains infinitesimally small. The figures presented throughout this thesis might increase their size for ease of viewing, but keep in mind that points are particular, incredibly tiny locations. One can hopefully understand why points are considered zero-dimensional objects, being infinitesimal in both length and width. Lines, alternatively, have infinite length and infinitesimal width (or vice versa); thus, lines are the one-dimensional analogs to zero-dimensional points.

Definition 1.1. Let P be a set of points in the (Euclidean) plane. An **ordinary line** is a line containing exactly two elements of P. A **connecting line** contains at least two elements of P.

It is apparent from these definitions that every ordinary line is a connecting line, but not every connecting line is an ordinary line. This nuance will play a key role in various results throughout the later sections. We now have the language to begin a discussion of incidence. The following postulates, albeit trivial, underpin our understanding of the subject:

Postulate 1.2. *A point is contained by an infinite number of lines.*

Postulate 1.3. *Two (distinct) points are contained by a unique line.*

There is no such postulate for a collection of three distinct points; the vertices of an equilateral triangle yield a counterexample. Thus, results pertaining to incidence become interesting when the collinearity of three or more distinct points is at play.

Definition 1.4. Let A, B, and C be not necessarily distinct points. If A, B, and C are contained by the same line, then they form the *collinear triple* denoted (ABC).

When the number of points in consideration increases, it becomes less and less obvious which collinear triples are formed, if any at all. Incidence theorems therefore provide a greater understanding of the relationship between sets of points and the lines that contain them. The results of these theorems are often not obvious, but provide

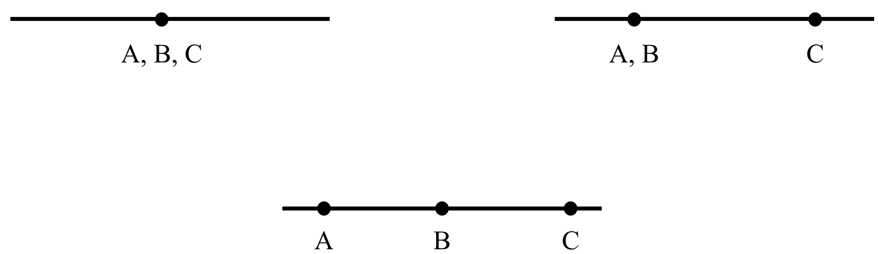

Figure 2. Up to labeling, there are three ways to arrange three points on a line. Postulate 1.2 applies to the left representation; Postulate 1.3 applies to the right.

profound insight into our understanding of various geometries. We will explore this connection in the upcoming chapters. Throughout this thesis, an incidence theorem will take the following form: *if some set of points forms some set of collinear triples, then at least one other collinear triple is also formed.* To approach the study of incidence theorems may seem daunting – there are an infinite number of possibilities to understand – but families of these theorems emerge quickly. Particularly, it is easy to categorize incidence theorems by their realizability. In the next section, we will introduce the relationship between geometric configurations and their underlying algebraic structures.

1.3. From Geometry to Algebra (and Back Again)

This thesis examines the realizability of various geometric configurations, a topic for which there are two ways of framing the problem. The first approach is visually intuitive and asks *"Where can we view these configurations?"* The second approach is equivalent, but much more abstract. It asks *"What algebraic structure is required to construct these objects?"* This thesis will repeatedly stress the importance of the connection between algebra and geometry illustrated by the study of configurations. So, how is realizability determined? Many strategies exist (some are better than others for different configurations), but it turns out that in many cases, the ways in which an incidence theorem can be proven informs the realizability of its corresponding geometric configuration. Given its rich history and large number of proofs, the famous Sylvester-Gallai Theorem exemplifies this phenomenon well. In this section, we will present two proofs of this theorem and unpack its realizability to better characterize the focus of this thesis. The theorem is stated below.

Theorem 1.5 (Sylvester-Gallai)**.** *Let P be a finite set of points in the plane, not all of which are collinear. Then P has an ordinary line.*

We note that the hypothesis of this theorem requires P to contain at least three distinct points in the plane, since every two points are collinear. An alternate (but equivalent) statement of the theorem says that if P is a finite set of points in the plane, then there exists a line that passes through either exactly two points in P

or every point in P. Many mathematicians have successfully provided proofs of this theorem, but M. Aigner and G. Ziegler describe Leroy Kelly's 1958 proof as "simply the best" in their anthology of the theorem's proofs [4]. Yes, this theorem is important enough to warrant such a text. Kelly's proof is incredibly straightforward and easy to visualize, earning its designation as best. We provide our version of this proof below.

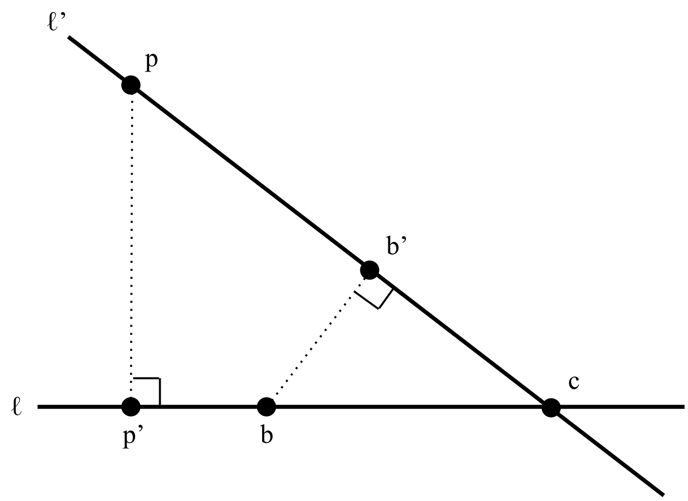

Figure 3. An illustration of Kelly's proof of the Sylvester-Gallai Theorem.

Kelly's Proof of Theorem 1.5 ([9]). For the sake of contradiction, suppose that every connecting line of points in P contains at least three points. Then there exist a point $p \in P$ and a connecting line ℓ such that $p \notin \ell$. Since P contains a finite number of points, there are choices for p and ℓ that are closer in terms of distance than all other such pairs of points and lines. An extension of Postulates 1.2 and 1.3 requires that there is a point $p' \in \ell$ such that the segment pp' is perpendicular to ℓ. Recall that ℓ contains at least three points in P by assumption: at least two of these points lie on one side of p'. Denote these points by b and c such that b lies between p' and c on ℓ. Note that b may coincide with p'. Postulate 1.3 guarantees the existence of the connecting line ℓ' which contains both p and c. As above, there is a point $b' \in \ell'$ such that the segment bb' is perpendicular to ℓ'. Then the triangles $T_1 = bb'c$ and $T_2 = pp'c$ are similar, with $T_1 \subseteq T_2$ due to our choices of b and b'. It follows that the length of bb' is less than that of pp', thereby violating our assumption that p and ℓ are the closest point-line pair. Then by contradiciton, ℓ is an ordinary line. □

Although tidy and concise, some mathematicians have criticized Kelly's usage of overcomplicated machinery; namely, the notions of Euclidean distance and perpendicularity. H. S. M. Coxeter, for example, describes this proof as "like using a sledge hammer to crack an almond." Geometers really do have a way with imagery. Coxeter asserted that "parallelism and distance are essentially foreign to this problem, which is concerned only with incidence and order." The proof which he provided (I like to imagine in aloof defiance) is provided below.

Coxeter's Proof of Theorem 1.5 ([6]). For the sake of contradiction, suppose that every connecting line of points in P contains at least three points and that not all points in P are collinear. It follows that some three of these points form a triangle in the plane. We denote these points by a, b, and c, and the triangle that they form by $T = abc$. Let L_a be the set of all connecting lines containing both a and at least one other element of P. If ℓ is not parallel to the line bc, then ℓ intersects bc. Suppose that $\ell \in L_a$ intersects bc at the point p' and that p' is not an element of P. Then if a connecting line of points in P is not parallel to ℓ, it meets ℓ either at a, p', or some other point. Travelling in the direction from a to p', denote the first of these intersection points that does not belong to the set P by p. Note that p may coincide with p'. As chosen, no connecting line of points in P meets ℓ beteween a and p. We will obtain a contradiction by proving the existence of such a connecting line.

Since $p \notin P$, we know that p lays on a connecting line containing at least three points in P. Denote these points by q, r, and s in such a way that q is between p and r, but s is not. Then since $a, r \in P$, we know that a third point $t \in P$ exists on the connecting line containing a and r. There are two cases to consider. First, suppose that t is between a and r. Then the the connecting line st intersects ℓ between a and p by Pasch's Theorem (a line that enters a triangle must exit the triangle) [6]. Next, suppose that t does not lie between a and r. Then the connecting line qt intersects ℓ betweeen a and p. Thus we reach a contradiction, so $T = abc$ cannot exist. □

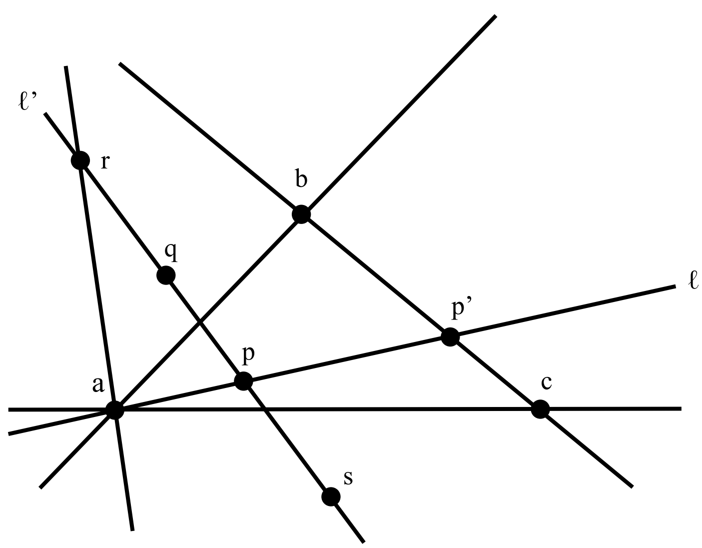

Figure 4. An illustration of Coxeter's proof of the Sylvester-Gallai Theorem. We place the point t somewhere on the line containing a and r.

While definitely more cumbersome, Coxeter's proof relies only on the notion of order, in contrast to the many properties required by Kelly's Proof. The difference between the proof styles exemplifies the realizability of the Sylvester-Gallai Theorem's corresponding geometric configuration. What we hope to show throughout this thesis is that fewer geometric properties required to prove an incidence theorem correspond to a more widely-realizable configuration. For example, Kelly's approach to proving the Sylvester-Gallai Theorem utilizes the concepts of length and angle (perpendicularity), both of which arise from axioms of Euclidean geometry, whereas Coxeter only uses ordering in his proof. Configurations that are realizable regardless of distance and angle are called *projective configurations*, as they can be realized in projective space. A full discussion of projective geometry will follow in the next chapter. It follows that since the proof of the Sylvester-Gallai Theorem requires ordering, then so too does the underlying algebraic structure of its corresponding projective configuration. Namely, we see that the configuration can be realized over the projective plane $\mathbb{C}P^2$, but not $\mathbb{R}P^2$. However, this conclusion makes little sense without knowledge of projective space, its axioms, and its transformations, so the next chapter will discuss these topics at length.

CHAPTER 2. FROM GEOMETRY TO ALGEBRA

The final paragraph of the previous chapter may have come across as nonsensical to those not well-versed in projective geometry. A space without distance or angle comes across rather abstractly. In this chapter, we hope to elucidate projective geometry as it pertains to our study of configurations and their realizability. We will focus particularly on two-dimensional projective planes, describing various methods for their construction and their underlying algebraic structures along the way. In short, we provide a fundamental connection between algebra and geometry.

2.1. The Projective Plane

Although it may appear daunting, the study of projective geometry – particularly that of the projective plane – is remarkably intuitive. Imagine yourself in the passenger seat of a car driving down a highway in Kansas. Perhaps the sun is setting in front of you; perhaps a certain cloud formation strikes your fancy; perhaps you just really like the highway. Whatever your motivation, you feel compelled to sketch the scene in front of you. As you draw, you notice that the boundaries of the highway begin to creep closer to one another as they approach the horizon, where they finally meet at a point. We know that these lines are parallel in actuality, of course. When we draw, though, we often take the idea of perspective into consideration absentmindedly. In order to convey the "three-dimensionality" of the world on a two-dimensional medium, parallel lines must meet somewhere. Surprise, dear reader, you're not in Kansas anymore. Welcome to the projective plane.

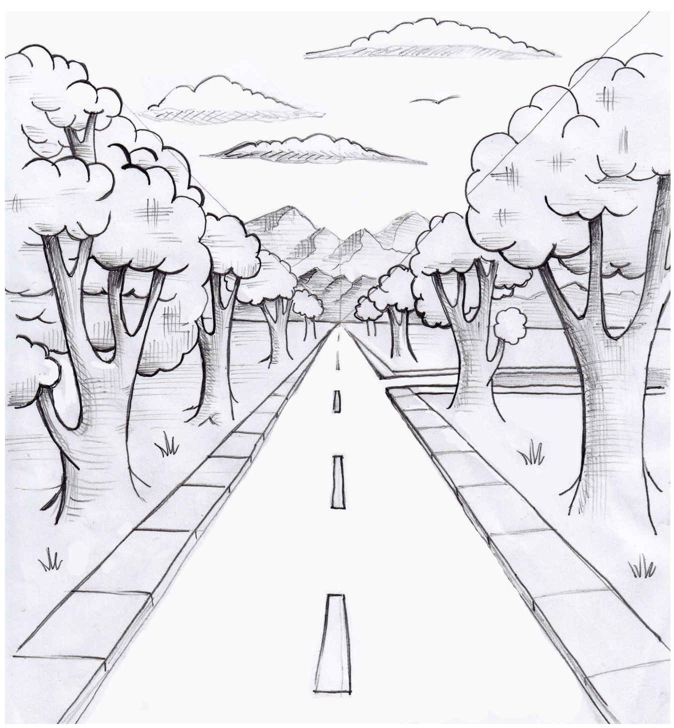

Figure 5. In this drawing, the parallel sides of the highway meet at a point.

Artists understood projective geometry centuries before its mathematical formalization. In fact, Gérard Desargues relied heavily on the concept of perspective in art to inform his studies while pioneering the subject in the early seventeenth century. He also noticed the phenomenon of the aforementioned highway drawing; that parallel lines in Euclidean space will intersect in projective space. We call such points of intersection *ideal points*, and the collection of all ideal points forms the *ideal line* in the projective plane. Intuitively, we can visualize an ideal point by "adding" it to the end of a line, so we often refer to ideal points as *points at infinity*. It follows that one-dimensional projective space is a circle. Formally, an n-dimensional projective space is defined as the union of a projective space of dimension $n-1$ with an n-dimensional Euclidean space. It follows that in two dimensions, the projective plane is formed by adjoining a line at infinity to the Euclidean plane. Mathematicians have successfully axiomized projective geometry by building upon this intuition, though we focus on the (two-dimensional) plane for the purposes of this thesis. A wide swath of projective planes exists, varying with respect to their underlying algebraic structure.

Definition 2.1. A *projective plane* $\mathcal{K}$ satisfies the following three axioms:

1. There is a unique line containing every two points in $\mathcal{K}$.
2. There is a unique intersection point for every two distinct lines in $\mathcal{K}$.
3. There are at least four points in $\mathcal{K}$ such that no three form a collinear triple.

The first axiom relates the two fundamental objects of geometry; the second, known as the elliptic parallel, serves as the crux of projective geometry; and the third provides a lower bound on the number of possible points in a finite projective plane. The astute reader will note that this definition suggests the existence of other axioms of projective geometry and will question their omission. The definition above provides the most general framework for a projective plane. When necessary, other axioms are assumed to help prove certain results. It turns out that these additional axioms help to restrict the possible underlying algebraic structures of projective planes, so at this point we recall some abstract algebra.

Definition 2.2. A *ring* $(R, +, \cdot)$ is an ordered triple such that $(R, +)$ is an Abelian group, that $\cdot$ is associative over R, and that $\cdot$ distributes over $+$ from both the left and the right for elements of R.

Definition 2.3. A *skew field* is a ring with the property that every nonzero element has a multiplicative inverse.

Definition 2.4. A *field* is a skew field such that $\cdot$ is commutative.

We are naturally familiar with a variety of number systems, most of which provide natural examples and counterexamples to the definitions above. The integers $\mathbb{Z}$ form a ring, for example, but the positive integers $\mathbb{Z}_+$ fail to satisfy this definition since they lack additive inverses. The quaternions create a skew field, but the "clock integers" $\mathbb{Z}/12\mathbb{Z}$ lack multiplicative inverses for every element: we see that $3k \not\equiv 1 \pmod{12}$ for every value of k. The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are both fields, just to name a few. In particular, the integers modulo p form the finite field $\mathbf{F}_p$ for every prime p. We hope to show that differences in underlying algebraic structures imply differences in projective planes, and vice versa.

2.2. Field-Based Projective Planes and Transformations

In the previous section, we discussed the axioms that all projective planes must satisfy and thereby (perhaps indirectly) outlined how to construct a projective plane axiomatically. We now shift our attention to a more direct construction rooted in algebra: the process of projectivizing a vector space. This method restricts the resulting projective planes to those defined over a field. With that being said, projectivizing a vector space allows us to gain great insights into the realizability of various configurations. In this section, we will outline the process of projectivization, then discuss the projective transformations that arise as a byproduct.

Let $\mathbf{V}$ be an n-dimensional vector space over a field $\mathbf{F}$. We "remove" the origin to consider $\mathbf{V}\setminus\{(0,\ldots,0)\}$ and define the equivalence relation $\sim$ such that $(x_1,\ldots,x_n) \sim (x'_1,\ldots,x'_n)$ if $(x_1,\ldots,x_n) = (kx'_1,\ldots,kx'_n)$ for some nonzero $k \in \mathbf{F}$. Defined formally, we projectivize this vector space as follows:

$$P\mathbf{F}^{n-1} := \mathbf{V} \setminus \{(0,\ldots,0)\} \big/ \sim$$

To gain a more geometric perspective, we note that lines through the origin in $\mathbf{V}$ correspond to points in $P\mathbf{F}^{n-1}$, and that planes through the origin in $\mathbf{V}$ correspond to lines in $P\mathbf{F}^{n-1}$. Generally, we say that m-dimensional objects through the origin in $\mathbf{V}$ correspond to $(m-1)$-dimensional objects in $P\mathbf{F}^{n-1}$. Thus, we say that the projective plane obtained by projectivizing a three-dimensional vector space $\mathbf{V}$ is the set of all lines containing the origin. This intuition allows us to understand that, while larger than $\mathbb{R}^2$, the projective plane is indeed two-dimensional.

It may seem difficult to visualize the projective plane abstractly, but its coordinatization should ease our understanding. Each point in the projective plane is assigned *homogenous coordinates*, which we denote using brackets and colons. For example, the point $(5,0,0)$ in $\mathbb{R}^3$ corresponds to the point $[5:0:0]$ in $P\mathbf{F}^2$. Due to the construction of $P\mathbf{F}^2$, though, we recognize that homogeonous coordinates are equivalence classes in the projective plane. Thus, the point $(5,0,0)$ also corresponds to $[1:0:0]$, $[-6:0:0]$, and $[\pi:0:0]$ in $P\mathbf{F}^2$ – these are different names for the same point of projective space. In fact, every point on the line $y = z = 0$ corresponds to $[1:0:0]$ in $P\mathbf{F}^2$. With this intuition, we can now better understand how to visualize the projective plane. It thus becomes clear that lines through the origin in 3-space correspond to points in the projective plane.

Suppose that $\mathbf{V} = \mathbb{R}^3$ for ease of visualization, although any three-dimensional vector space will suffice, and let $p = (x,y,z)$ be a nonzero point in $\mathbb{R}^3$. Then there is a vector $\vec{v}$ pointing from the origin to p, defined explicitly by $\vec{v} = \langle x,y,z\rangle$. Then the corresponding unit vector $\hat{v} = \vec{v}/\|\vec{v}\|$ points from the origin to a point on the unit sphere in $\mathbb{R}^3$ for all $p \neq (0,0,0)$. We denote this point by p' and obtain homogenous coordinates for p by writing the components of p' with brackets and colons. It should be noted that these homogenous coordinates are simply representatives of their respective equivalence classes, though assigning them unit length in $\mathbb{R}^3$ allows a more standardized understanding.

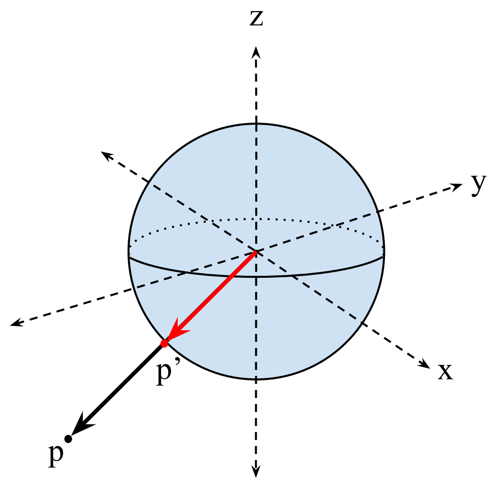

Figure 6. We obtain homogenous coordinates in $\mathbb{R}P^2$ by computing p' from p.

Proposition 2.5. *Let* $\boldsymbol{V}$ *be a three-dimensional vector space defined over a field* $\boldsymbol{F}$*. Then* $P\mathbf{F}^2$ *satisfies Definition 2.1.*

Proof. Since $\mathbf{V}$ is a vector space, there is a unique plane that contains every two distinct lines through the origin in $\mathbf{V}$. Thus there is a unique line containing every two points in $P\mathbf{F}^2$. Similarly, we know that every two planes containing the origin in $\mathbf{V}$ intersect at a line through the origin. It follows that every two lines in $P\mathbf{F}^2$ intersect at a point. Finally, note that 0 and 1 are distinct elements of $\mathbf{F}$ since $\mathbf{F}$ is a field. Then $[0:1:0]$, $[1:0:0]$, $[1:0:1]$, and $[0:1:1]$ are elements of $P\mathbf{F}^2$ such that no three are collinear. $\square$

At this point in the chapter, the reader might ask "but why?" In short: algebraic constructions yield geometric generalizations. Recall that in Chapter 1.1, we stated that Monsky's Theorem holds over any rectangularly-bounded region, though our example only involved one set of dimensions. It turns out that this generalization arises as a byproduct of the *affine transformations* – dilation, rotation, translation, and reflection – which we may apply since the rectangle in question is embedded in the affine plane. The "big picture" idea of this section is that projectivizing a vector space over some field allows us to make use of projective transformations in the subsequent chapters for ease of our proofs. We define and explore the projective transformations below.

Definition 2.6. Let $\mathbf{F}$ be a field and consider $P\mathbf{F}^2$. If a map $T : P\mathbf{F}^2 \to P\mathbf{F}^2$ preserves every collinear triple, then T is a *projective transformation*.

What do these transformations actually look like? Here, we exploit the definition of projectivization to categorize these maps. In linear algebra, we learned that every invertible 3×3 matrix M (over $\mathbf{F}$) acts on $\mathbf{F}^3$ by sending lines through the origin to lines through the origin. These matrices are elements of the general linear group $GL_3(\mathbf{F})$. As mentioned above, lines in $\mathbf{F}^3$ correspond with points in $P\mathbf{F}^2$ under projectivization. Thus, M acts on $P\mathbf{F}^2$ by mapping points to points. It can be shown that M is incidence-preserving and is therefore a projective transformation. Further, we recall that projectivization is invariant under scalar multiplication. For every nonzero $k \in \mathbf{F}$, then, we know that kM is invertible and induces the same map as M on $P\mathbf{F}^2$. When defined up to scalar equivalence, these matrices are elements of the projective general linear group $PGL_3(\mathbf{F})$. Formally, $PGL_3(\mathbf{F})$ is the quotient of $GL_3(\mathbf{F})$ by its center $\{kI_3 : k \in \mathbf{F} \setminus \{0\}\}$. For the purposes of this thesis, we restrict our attention to only these transformations.

Theorem 2.7. *Every element of $PGL_3(\boldsymbol{F})$ is a projective transformation.*

Just as the projective plane extends the affine plane, the group of projective transformations extends the group of affine transformations $\mathrm{Aff}_2(\mathbf{F})$. Recall that an affine transformation f is a combination of rotations, dilations, translations, and reflections. Algebraically, we say that f is some composition of an invertible 2×2 matrix A and addition of a vector v. Extending f to a projective transformation T requires that we define it appropriately on the ideal line. The embedding $\sigma : \mathrm{Aff}_2(\mathbf{F}) \to PGL_3(\mathbf{F})$ explicitly realizes this extension, where $\sigma(f) = T$. A more formal definition of this embedding is found in Remark 2.8 below.

Remark 2.8. *If* $f\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} + v$, *then* $\sigma(f) = T$ *is the* 3×3 *matrix* $\begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix}$.

Understanding the group of projective transformations allows us to better comprehend the behavior of points and lines in the projective plane. We now provide a series of results that enrich our study and ameliorate subsequent proofs.

Lemma 2.9. *Let $p, q, r \in P\mathbf{F}^2$ such that $p = [p_1 : p_2 : p_3]$, $q = [q_1 : q_2 : q_3]$, and $r = [r_1 : r_2 : r_3]$. Then p, q, r are collinear if and only if the following equation holds:*

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = 0.$$

Proof. Note that p, q, r are collinear if and only if the lines $\ell_p, \ell_q, \ell_r \in \mathbf{F}^3$ are coplanar, where each ℓ_i contains the origin and corresponds to the point i upon the projectivization of $\mathbf{F}^3$. Then ℓ_p, ℓ_q, ℓ_r are linearly dependent over $\mathbf{F}$ and so $\det[\ell_p \ell_q \ell_r] = 0$. Since determinants are preserved through transpose, we also have $\det[\ell_p \ell_q \ell_r]^T = 0$. It follows that $\det[pqr] = \det[pqr]^T = 0$, thereby proving the given statement. □

Lemma 2.10. *Let $p, q, r, s \in P\mathbf{F}^2$ be distinct and such that no three are collinear. Then there exists a unique $f \in PGL_3(\boldsymbol{F})$ such that $f(p) = [1:0:0]$, $f(q) = [0:1:0]$, $f(r) = [0:0:1]$, and $f(s) = [1:1:1]$ [10].*

Proof. Let $f^{-1} \in PGL_3(\mathbf{F})$ correspond to the matrix $A = [a_{ij}]$ where $1 \le i, j \le 3$. Then there is some nonzero $k_1 \in \mathbf{F}$ such that $k_1 p = A \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = [a_{11}\ a_{12}\ a_{13}]$. Thus, we have determined the first row of A up to some nonzero scaling factor. Similar logic allows us to define the second and third rows up to nonzero scaling factors $k_2, k_3 \in \mathbf{F}$. Explicitly, the second and third rows are $k_2 q$ and $k_3 r$, respectively. It follows that $s = A \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ if and only if there is some nonzero $k_4 \in \mathbf{F}$ such that $k_4 s = k_1 p + k_2 q + k_3 r$. We re-scale this equation in such a way that $k_4 = 1$. Then since p, q, r are linearly independent, this system of three equations in three unknown variables has a unique solution (k_1, k_2, k_3). We note that since k_4 is independent from any two other k_i, the matrix A is invertible and thus defines a projective transformation. We also note that A is unique up to scale [10]. □

Corollary 2.11. *Let $p, q, r \in P\mathbf{F}^2$ be distinct. Then p, q, r are non-collinear if and only if some $f \in PGL_3(\mathbf{F})$ satisfies $f(p) = [0:0:1]$, $f(q) = [1:0:0]$, and $f(r) = [0:1:0]$.*

Corollary 2.12. *Let $p, q, r \in P\mathbf{F}^2$ be distinct. Then p, q, r are collinear if and only if some $f \in PGL_3(\mathbf{F})$ satisfies $f(p) = [1:0:0]$, $f(q) = [1:0:1]$, and $f(r) = [0:0:1]$.*

Proof. Let $p = [p_1 : p_2 : p_3]$, $q = [q_1 : q_2 : q_3]$, and $r = [r_1 : r_2 : r_3]$, and suppose that p, q, r are collinear. Then since p, q, r are distinct, we may write r as a linear combination of p and q. Now, we consider the matrix

$$A^{-1} = \begin{bmatrix} k_1 p_1 & k_1 p_2 & k_1 p_3 \\ x_1 & x_2 & x_3 \\ k_2 r_1 & k_2 r_2 & k_2 r_3 \end{bmatrix},$$

where k_1, k_2 are nonzero elements of $\mathbf{F}$ and $x = [x_1 : x_2 : x_3]$ is linearly independent from $k_1 p$ and $k_2 r$. Thus, A^{-1} is an invertible 3×3 matrix and is thereby a projective transformation; so too is its inverse. Simple computations reveal that:

$$A^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = k_1 p, \quad A^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = k_2 r, \quad A^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = k_1 p + k_2 r.$$

It follows that the matrix A is the projective transformation that we desire. Now, suppose that some $f \in PGL_3(\mathbf{F})$ satisfies $f(p) = [1:0:0]$, $f(q) = [1:0:1]$, and

$f(r) = [0 : 0 : 1]$. Then f corresponds to some invertible matrix M and we write:

$$M^{-1} = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix}$$

where each $m_i \in \mathbf{F}$. The following results follow naturally:

$$\begin{aligned} p &= [m_1 : m_2 : m_3] \\ q &= [m_1 + m_7 : m_2 + m_8 : m_3 + m_9] \\ r &= [m_7 : m_8 : m_9] \end{aligned}$$

Then q is a linear combination of p and r, so p, q, r are collinear. □

Corollary 2.12 can be extended to show generally that there is a projective transformation that maps any three distinct and collinear points to any other three distinct and collinear points. Similarly, Corollary 2.11 shows that there is a projective transformation that maps any three distinct and non-collinear points to any other three distinct and non-collinear points. These results allow us to better comprehend how points and lines behave in the projective plane. For much of the remainder of this thesis, we will apply these transformations to ameliorate our proofs.

2.3. Configurations and Realizability

With a more solid understanding of the projective plane and its associated group of transformations, we may (finally) begin a formal discussion of projective configurations and their realizability. Below, we provide some preliminary definitions.

Definition 2.13. A *configuration* is a pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a set of points and $\mathcal{L}$ is a set of triples generated from points in $\mathcal{P}$. A *projective configuration* can be realized in at least one projective plane.

In other words, a configuration is an abstract "desire" of a certain geometry where the points and lines have been prescribed. Since $\mathcal{L}$ is made up of triples, the definition only makes sense if $(\mathcal{P}, \mathcal{L})$ has a realization in the projective plane.

Definition 2.14. A *realization* of a projective configuration $(\mathcal{P}, \mathcal{L})$ in $P\mathbf{F}^2$ is a map $\rho : \mathcal{P} \to P\mathbf{F}^2$ such that for every $\ell \in \mathcal{L}$, the image $\rho(\ell)$ is collinear.

Explicitly, realizations map abstract triples to collinear triples in the projective plane. There are many different ways to realize a configuration, though.

Definition 2.15. A realization ρ is *combinatorially complete* if it is injective. Otherwise, the realization is *combinatorially degenerate*. If the image $\rho(\mathcal{P})$ is contained by a single line, then the realization is *fully collinear*.

Thus, there are four flavors of realizability for us to consider. A realization can be both combinatorially complete and fully collinear, for example. Before launching into that discussion, though, we conclude this chapter with two motivating examples of projective configurations and the realizability.

Lemma 2.16 (General Tetrahedron)**.** *Let* $(\mathcal{P}, \mathcal{L})$ *be a projective configuration where* $\mathcal{P} = \{A, B, C, D\}$ *and* $\mathcal{L} = \{(ABC), (ABD), (BCD)\}$*. Any realization of* $(\mathcal{P}, \mathcal{L})$ *is fully collinear.*

Proof. For the sake of contradiction, suppose that a realization does not form (ACD), so A, C, and D determine the vertices of a non-degenerate triangle. There are two cases to consider for the position of B, without loss of generality. First, suppose that $A = B$. Then (BCD) implies (ACD), yielding a contradiction. Next, suppose that $A \neq B$. Then there is a line ℓ containing both A and B. It follows from (ABC) that $C \in \ell$ and from (BCD) that $D \in \ell$. Thus, A, B, D are collinear and we reach a contradiction. We conclude that the realization must form (ACD) and thereby that the realization is fully collinear. □

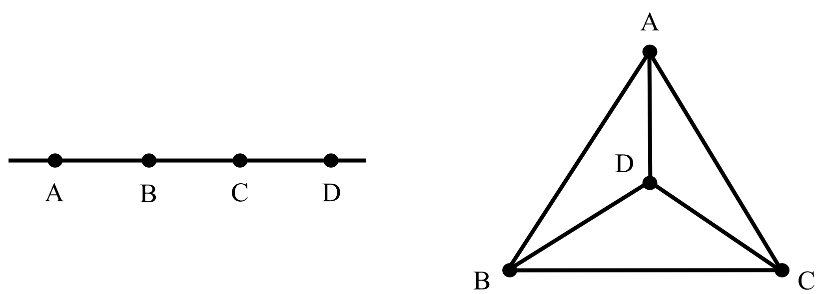

Figure 7. An illustration of Lemma 2.16. Every realization must be fully collinear.

Corollary 2.17 (Distinct Tetrahedron)**.** *Let* $(\mathcal{P}, \mathcal{L})$ *be a projective configuration where* $\mathcal{P} = \{A, B, C, D\}$ *and* $\mathcal{L} = \{(ABC), (BCD)\}$*. If a realization of* $(\mathcal{P}, \mathcal{L})$ *is combinatorially complete, then it is fully collinear.*

Since they only contain four points, the above configurations are perhaps the easiest to visualize and comprehend. In each case, full collinearity is attained independent of field; that is, coordinates are not required to prove these results.

Definition 2.18. A projective configuration is *universally realizable* if it can be realized over every projective plane.

There are many configurations that are not universally realizable, though, and the majority of this thesis will investigate such configurations. Many canonical results of projective geometry correspond to not universally realizable configurations, for example. One of these famous theorems is that of Desargues, which we will show introduces some additional structure to Definition 2.1. It is by nature an incidence theorem and is presented below.

Theorem 2.19 (Desargues's Theorem)**.** *Let A, B, C, D, E, F, O be distinct points in a projective plane such that (ADO), (BEO), and (CFO) are collinear triples. Then there exists a line ℓ on which the following pairs of lines intersect: AC and DF; BC and EF; and AB and DE.*

Some visual clarity is provided in Figure 8. Desargues's Theorem holds in any projective space with dimension greater than or equal to 3, but there are many flavors of the projective plane over which this theorem fails: we refer to these planes as *non-Desarguesian*. If a projective plane is *Desarguesian*, then it can be assigned coordinates over some skew field. In the context of this thesis, assuming the validity of Desargues's Theorem allows us to conclude that various projective configurations are realizable over skew fields. It follows that a projective configuration corresponding to the assumptions of Desargues's Theorem is not univerally realizable. In the subsequent chapters, we will make expressively clear the cases in which this assumption is necessary. It should be noted that in the vast majority of extant literature on projective geometry (e.g. [5, 8]), Desargues's Theorem is discussed in close proximity to the theorem of Pappus, another wildly influential result. For example, Hilbert (somewhat inaccurately) states that "any theorems concerned solely with incidence relations in the [Euclidean projective] plane can be derived from [Pappus' Theorem]" [8].

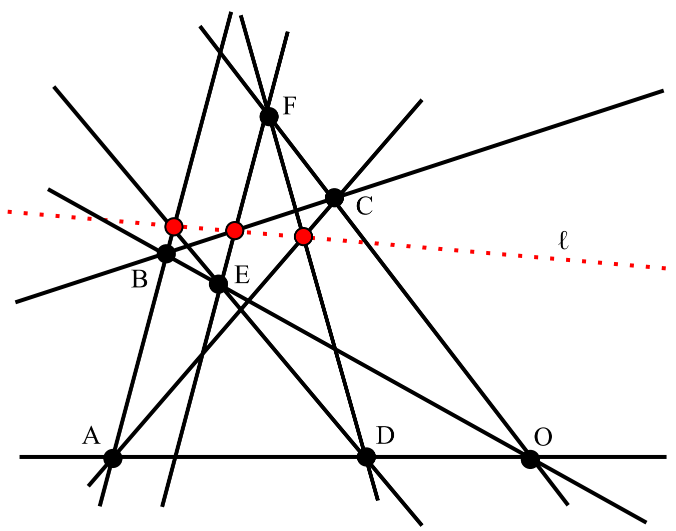

Figure 8. An illustration of Desargues's Theorem, which only holds over skew fields.

Theorem 2.20 (Pappus's Theorem)**.** *Let $\mathcal{P} = \{A, B, C, D, E, F, G, H, I\}$ be a set of points in the projective plane. If $\mathcal{P}$ forms the collinear triples (ABC), (ADH), (AEI), (BDG), (BFI), (CEG), (CFH), and (GHI), then $\mathcal{P}$ induces (DEF).*

As with Desargues's Theorem, it turns out that Pappus's Theorem only holds over specific types of projective planes. Specifically, a *Pappian plane* can be assigned

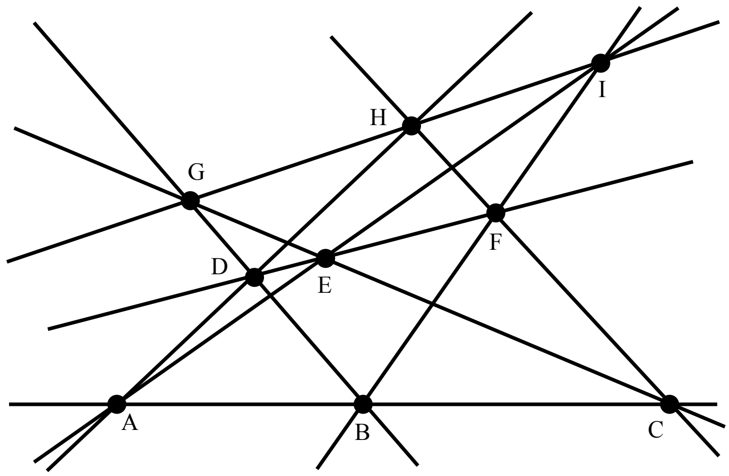

Figure 9. An illustration of Pappus's Theorem, which only holds over fields.

coordinates over some field. Any projective configuration corresponding to the assumptions of Pappus's Theorem is thereby not universally realizable. There are many other axioms of projective geometry that prove useful for our studies. Marchisotto proved that by adding two additional axioms to our definition of a projective plane, we restrict our focus to those which extend a field [11]. The first of these axioms is that of Fano, which states that the three diagonal points of a complete quadrangle are never collinear. Assuming this axiom also requires that these projective planes' underlying algebraic structures cannot have characteristic 2. The Fano plane thus does not satisfy this axiom, for example, and so the Fano configuration (discussed subsequently in Chapter 3.1) is thereby not realizable in this context. Marchisotto's final assumed axiom requires that if a projective transformation leaves three points on a line invariant, it leaves every point on that line invariant. Marchisotto goes on to prove that assuming these axioms restricts our focus to Pappian planes alone, illustrating a powerful connection between algebra and geometry. This proof will be discussed in more detail in Chapter 4.

Just as all fields are by definition skew fields, so too are all Pappian planes by definition Desarguesian. In other words, Pappus's Theorem implies Desargues's. For the purposes of this thesis, Pappian planes allow us to make sweeping generalizations about the realizability of various projective configurations, mostly due to the convenience of their coordinatization. If a configuration is not universally realizable, assigning coordinates to its realization often ameliorates the proof. Thus, the results of Lemma 2.10, Corollary 2.12, and Corollary 2.11 will greatly ease our understanding. With this in mind, we will investigate the different flavors of realizability for a variety of configurations in the subsequent chapters.

CHAPTER 3. SYMMETRIC PROJECTIVE CONFIGURATIONS

The central purpose of this thesis is to investigate the relationships between the constructions of various projective configurations and the algebraic structures underlying the planes over which they are realizable. If that sentence intimidates you, dear reader, fret not. This topic seems insurmountable at first, but there are more easily accessible avenues into this study which we hope to use as an entry into a larger discussion of realizability. In this chapter, we will investigate *symmetric configurations*, whose various symmetries allow us to make several generalizations due to the properties of projective transformations (discussed in the previous chapter), which in turn ameliorate the examination of their realizability. We say that a projective configuration is symmetric if each point lies on the same number of lines and every two points are joined by exactly one line. The configurations corresponding to the theorems of Desargues and Pappus, for example, are both symmetric in nature. This chapter will focus primarily on two such configurations, the Fano configuration and the Hesse configuration, as gentle and accessible introductions to our study.

3.1. The Fano Configuration

In Chapter 2.2, we outlined the process used to projectivize any given field. We note that the smallest projective plane is the projective Fano plane $P(\mathbf{F}_2)^2$, constructed over $\mathbf{F}_2$, the field with two elements. This section will examine the corresponding Fano configuration and discuss its realizability. In the process, we hope to make more clear the connection between projective configurations and the underlying algebraic structures of the fields over which they may be realized.

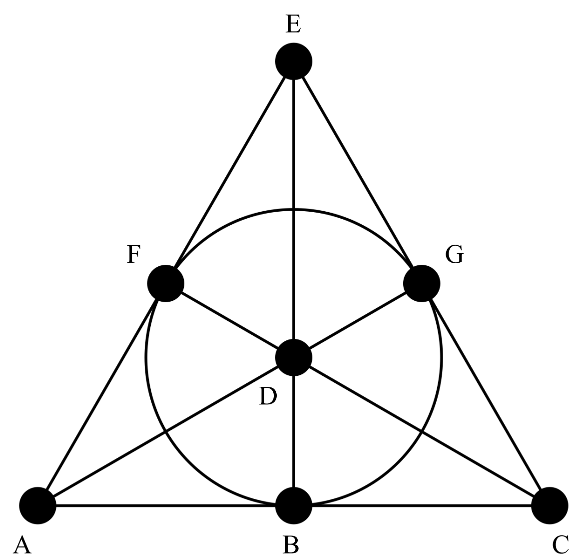

Figure 10. A realization of the Fano configuration in the projective Fano plane $P(\mathbf{F}_2)^2$.

Definition 3.1. The *Fano configuration* is a configuration $\mathcal{F} = (\mathcal{P}, \mathcal{L})$ such that $\mathcal{P} = \{A, B, C, D, E, F, G\}$ and $\mathcal{L} = \{(ABC), (ADG), (AEF), (BFG), (BDE), (CDF), (CEG)\}$.

Remark 3.2. *Each point in $\mathcal{F}$ lies on exactly three lines.*

Remark 3.3. *Exactly one line joins every two points in $\mathcal{F}$.*

It follows quickly from these remarks that the Fano configuration is symmetric. Indeed, mathematicians studying incidence geometry (e.g. [5, 8]) often refer to the Fano configuration as 7_3, where the numeral 7 denotes the number of points contained in $\mathcal{F}$ and the subscript 3 describes the number of lines passing through each point. It is worth emphasizing that this notation only applies to symmetric configurations. In the next section, we will discuss the Hesse configuration 9_4 and its connections with 8_3. First, though, we will discuss the realizability of the Fano configuration.

Theorem 3.4. *Let $\mathbf{F}$ be a field. The Fano configuration has a combinatorially complete realization over $P\mathbf{F}^2$ if and only if $\mathbf{F}$ has characteristic 2.*

Proof. Suppose $\mathcal{F}$ has a combinatorially complete realiztion over $P\mathbf{F}^2$. Lemma 2.10 allows us to assume that $A = [1 : 0 : 0]$, $B = [0 : 1 : 0]$, $D = [0 : 0 : 1]$, and $F = [1 : 1 : 1]$. Now write that $C = [c_1 : c_2 : c_3]$. Since (ABC) is collinear, Lemma 2.9 requires that the following equation holds:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = c_3 = 0.$$

A similar computation arising from (CDF) yields that $c_1 = c_2$. Thus, we write that $C = [c_1 : c_1 : 0]$. Since c_1 must be nonzero, we re-scale so that $C = [1 : 1 : 0]$ Nearly identical processes help us to locate E and G. First, write that $E = [e_1 : e_2 : e_3]$. Then since (BDE) is collinear, Lemma 2.9 requires that the following equation holds:

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ e_1 & e_2 & e_3 \end{vmatrix} = e_1 = 0.$$

The computation arising from (AEF) yields that $e_2 = e_3$, and so we write that $E = [0 : e_2 : e_2]$. Since e_2 must be nonzero, we re-scale so that $E = [0 : 1 : 1]$. Next, we write that $G = [g_1 : g_2 : g_3]$. Applying Lemma 2.9 to the collinear triples (ADG) and (BFG) determines that $g_2 = 0$ and that $g_1 = g_3$, respectively. Thus, $G = [g_1 : 0 : g_1]$. Since g_1 must be nonzero, we re-scale so that $G = [1 : 0 : 1]$ Finally, we apply Lemma 2.9 to (CEG) to yield the following equation:

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1 + 1 = 0.$$

Then **F** has characteristic 2 by definition. Now, suppose that **F** has characteristic 2. Then we may realize $\mathcal{F}$ in the following coordinates:

$$A = [1 : 0 : 0]$$
$$B = [0 : 1 : 0]$$
$$C = [1 : 1 : 0]$$
$$D = [0 : 0 : 1]$$
$$E = [0 : 1 : 1]$$
$$F = [1 : 1 : 1]$$
$$G = [1 : 0 : 1]$$

This realization of $\mathcal{F}$ is combinatorially complete. □

3.2. The Hesse Configuration

At the end of Chapter 1.3, we discussed the Sylvester-Gallai Theorem and its corresponding geometric configurations. In this section, we will examine at length the Hesse configuration, perhaps the best-known counter-example to the Sylvester-Gallai Theorem, discussing in the process its combinatorial degeneracies and realizability.

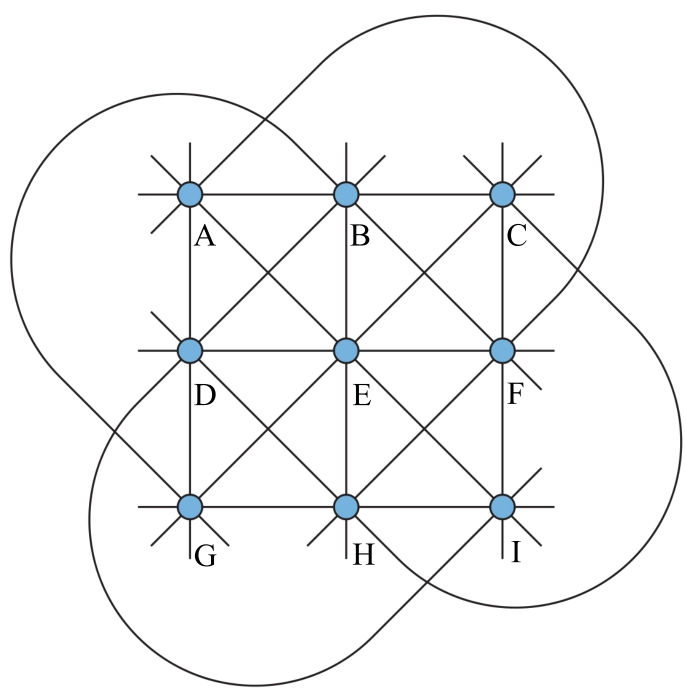

Figure 11. A schematic of a combinatorially complete realization of the Hesse configuration. It can be realized over the projective plane $P\mathbb{C}^2$, but not over $P\mathbb{R}^2$.

Definition 3.5. The *Hesse configuration* is a configuration $\mathcal{H} = (\mathcal{P}, \mathcal{L})$ such that $\mathcal{P} = \{A, B, C, D, E, F, G, H, I\}$ and $\mathcal{L} = \{(ABC), (BDI), (CEG), (ADG), (BEH), (CFI), (AFH), (BFG), (DEF), (AEI), (CDH), (GHI)\}$.

Though perhaps unclear from its definition alone, the Hesse configuration is indeed symmetric. The following remarks outline its symmetries in a condensed fashion; these results will aid in our understanding of the configuration and play a large role in several upcoming proofs.

Remark 3.6. *Each point in $\mathcal{H}$ lies on exactly four lines.*

Remark 3.7. *Exactly one line joins every two points in $\mathcal{H}$.*

As noted in the previous section, many texts (e.g. [5, 8]) refer to the Hesse configuration as 9_4. There is a strange connection between this configuration and 8_3; namely, in the fields over which they are realizable. Previous mathematicians have investigated the realizability of the combinatorially complete Hesse configuration, but no known sources have examined its combinatorial degeneracies. Throughout this section, we aim to investigate every combinatorially-possible Hesse configuration to provide an anthology of its realizability. We begin by addressing the realizability of its combinatorially complete configuration. While the following result has been stated previously, a satisfactory proof remains absent in the source material. We address this issue below.

Theorem 3.8. *A combinatorially complete realization of $\mathcal{H}$ exists over $P\mathbf{F}^2$ if and only if some $x \in \mathbf{F}$ satisfies the equation $x^2 - x + 1 = 0$.*

Proof. Suppose that a realization of $\mathcal{H}$ in $P\mathbf{F}^2$ is combinatorially complete. We assume that $A = [1 : 0 : 0]$, $B = [0 : 1 : 0]$, $D = [0 : 0 : 1]$, and $E = [1 : 1 : 1]$ by Lemma 2.10. Now write that $I = [i_1 : i_2 : i_3]$. Since (BDI) is collinear, Lemma 2.9 requires that the following equation holds:

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ i_1 & i_2 & i_3 \end{vmatrix} = -i_1 = 0.$$

A similar computation arising from (AEI) yields that $i_2 = i_3$. Thus, we re-scale to write that $I = [0 : 1 : 1]$. We use nearly identical methods to locate the points C, F, G, and H. First, write that $C = [c_1 : c_2 : c_3]$, $F = [f_1 : f_2 : f_3]$, $G = [g_1 : g_2 : g_3]$, and $H = [h_1 : h_2 : h_3]$. Applying Lemma 2.9 to (ABC) yields that $c_3 = 0$. Similarly, Lemma 2.9 requires that $g_2 = 0$ from (ADG), that $f_1 = f_2$ from (DEF), and that $h_1 = h_3$ from (BEH). Thus, we have the following:

$$C = [c_1 : c_2 : 0] \qquad F = [f_1 : f_1 : f_3] \qquad G = [g_1 : 0 : g_3] \qquad H = [h_1 : h_2 : h_1]$$

Since the realization of $\mathcal{H}$ is combinatorially complete, we know that $F \neq D$. It follows that $f_1 \neq 0$, so we re-scale and write $F = [1 : 1 : f^*]$. By the same logic, we know that $H \neq B$. Then $h_1 \neq 0$, so we re-scale and write $H = [1 : h^* : 1]$. Next, we apply Lemma 2.9 to (CDH), yielding that $c_2 = c_1 h^*$. We note now that if $c_1 = 0$, then $C = [0 : 0 : 0]$ which is impossible. It follows that $c_1 \neq 0$ and re-scaling allows us to write that $C = [1 : h^* : 0]$. Similarly, we apply Lemma 2.9 to (BFG) and discern that $g_3 = g_1 f^*$. We see that if $g_1 = 0$, then $G = [0 : 0 : 0]$ which is impossible. It follows that $g_1 \neq 0$ and re-scaling allows us to write that $G = [1 : 0 : f^*]$. For the sake of clarity, we now have the following:

$$C = [1 : h^* : 0] \qquad F = [1 : 1 : f^*] \qquad G = [1 : 0 : f^*] \qquad H = [1 : h^* : 1]$$

At this point, there are four collinear triples that we have not yet applied Lemma 2.9 to: (CEG), (CFI), (GHI), and (AFH). For the first three of these four, applying Lemma 2.9 shows that $h^* + f^* = 1$. Applying the lemma to (AFH) requires that $h^* f^* = 1$. We will now manipulate these equations to prove the statement in the forward direction.

First, recall that the realization of $\mathcal{H}$ is combinatorially complete. Thus, $C \neq A$ and $G \neq A$. It follows from these constraints that $h^* \neq 0$ and $f^* \neq 0$, respectively. We now combine the two equations to yield $h^* + f^* = h^* f^*$. Then we have

$$\begin{aligned} f^* &= 1 + f^*/h^* \\ f^* &= (h^* f^*) + f^*/h^* \\ f^* h^* &= f^*(h^*)^2 + f^* \\ h^* &= (h^*)^2 + 1 \end{aligned}$$

We conclude from this equation that $0 = (h^*)^2 - h^* + 1$. Now, suppose that some $x \in \mathbf{F}$ satisfies $x^2 - x + 1 = 0$. Then we may realize $\mathcal{H}$ in the following coordinates:

$$\begin{aligned} A &= [1 : 0 : 0] \\ B &= [0 : 1 : 0] \\ C &= [1 : x : 0] \\ D &= [0 : 0 : 1] \\ E &= [1 : 1 : 1] \\ F &= [1 : 1 : 1 - x] \\ G &= [1 : 0 : 1 - x] \\ H &= [1 : x : 1] \\ I &= [0 : 1 : 1] \end{aligned}$$

This realization of $\mathcal{H}$ is combinatorially complete. □

Corollary 3.9. *A combinatorially complete realization of $\mathcal{H}$ exists in $P\mathbb{C}^2$. Such a realization does not exist in $P\mathbb{R}^2$.*

We now use this result in an attempt to explicity outline over which fields the combinatorially complete Hesse configuration is realizable. While rather tangential for the purposes of this thesis, additional information can be found in [13]. Suppose that $\mathbf{F}$ is a field with characteristic greater than 3. Then the quadratic formula guarantees that $x^2 - x + 1 = 0$ has a solution over $\mathbf{F}$ if and only if $\sqrt{-3} \in \mathbf{F}$. Equivalently, we know that -3 must be a square in $\mathbf{F}$. Through arguments of quadratic reciprocity, we may characterize the finite, prime fields that satisfy such a property.

Lemma 3.10. *If $\boldsymbol{F}_p$ is a finite field with $p = 3$, then $x = \bar{2}$ satisfies $x^2 - x + 1 = 0$.*

Lemma 3.11. *Let $\boldsymbol{F}_p$ be a finite field with $p > 3$. Some $x \in \boldsymbol{F}_p$ satisfies the congruence $x^2 \equiv -3 \pmod{p}$ if and only if $p \equiv 1 \pmod{6}$.*

Proof. We make use of the Legendre symbol and quadratic reciprocity [13]. By definition, there exists an $x \in \mathbf{F}_p$ satisfying $x^2 \equiv -3 \pmod{p}$ if and only if we have:

$$\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = 1.$$

There are two cases to consider in satisfying this equation.

1. First, suppose that $(\frac{-1}{p}) = (\frac{3}{p}) = 1$. We know that if $(\frac{-1}{p}) = 1$, then $p \equiv 1 \pmod{4}$. We thereby obtain through reciprocity that $(\frac{3}{p}) = (\frac{p}{3})$. By definition, we know that $(\frac{p}{3}) = 1$ if $p \equiv x^2 \pmod{3}$ for some $x \in \mathbf{F}_p$. We note now that $2 \equiv x^2 \pmod{3}$ has no solutions and that $p \not\equiv 0 \pmod{3}$ since p is a prime greater than 3. Thus, we need only consider the case where $p \equiv 1 \pmod{3}$, which occurs when $p \equiv 1 \pmod{12}$.

2. Now, suppose that $(\frac{-1}{p}) = (\frac{3}{p}) = -1$. If $(\frac{-1}{p}) = -1$, then $p \equiv 3 \pmod{4}$. It follows through reciprocity that $(\frac{3}{p}) = -(\frac{p}{3})$, so once again we investigate the cases in which $(\frac{p}{3}) = 1$. We need only consider the case where $p \equiv 1 \pmod{3}$, so $p \equiv 7 \pmod{12}$.

The only relevant cases occur when $p \equiv 1 \pmod{12}$ or $p \equiv 7 \pmod{12}$; equivalently, we may write that $p \equiv 1 \pmod{6}$. □

Lemma 3.12. *A finite field $\mathbf{F}$ contains a root to $x^2 - x + 1 = 0$ if and only if it has characteristic $p = 3$, $p \equiv 1 \pmod{6}$, or is an even-degree extension over $\mathbf{F}_p$.*

Proof. Let $\mathbf{F}$ be a finite field over $\mathbf{F}_p$, and let $f(x) = x^2 + x + 1 \in \mathbf{F}_p[x]$. Suppose that $\mathbf{F}$ contains a root ω of f. If $f(x)$ is reducible, then $\mathbf{F}_p$ contains a root of f, and if the characteristic of $\mathbf{F}$ is not 2, then $\mathbf{F}_p$ contains $\sqrt{-3}$. It follows from Lemma 3.11 that either $p = 3$ or $p \equiv 1 \pmod{6}$. Otherwise, $f(x)$ is irreducible. Since ω is a root of f over $\mathbf{F}_p$, it follows that $[\mathbf{F}_p(\omega) : \mathbf{F}_p] = 2$. Then we know that

$$[\mathbf{F} : \mathbf{F}_p] = [\mathbf{F} : \mathbf{F}_p(\omega)][\mathbf{F}_p(\omega) : \mathbf{F}_p].$$

It follows that $\mathbf{F}$ is an extension of even degree over $\mathbf{F}_p$. In other words, we conclude that $\mathbf{F} = \mathbf{F}_{p^k}$, where k is even.

Suppose now that $p = 3$ or $p \equiv 1 \pmod 6$. Then Lemmas 3.10 and 3.11 guarantee the existence of a root ω of f in $\mathbf{F}_p$. If $\mathbf{F}_p$ does not contain such a root, then $x^2 + x + 1$ is irreducible over $\mathbf{F}_p$ and thus $K := \mathbf{F}_p[x]/(x^2 + x + 1)$ is a field. It follows that $[K : \mathbf{F}_p] = 2$, so $K = \mathbf{F}_{p^2}$ for some odd prime p. Now, suppose that $\mathbf{F}$ is an even-degree extension over $\mathbf{F}_p$, so $\mathbf{F} = \mathbf{F}_{p^k}$ for some even k. We know that $\mathbf{F}_{p^m} \subseteq \mathbf{F}_{p^n}$ if and only if $m|n$. Since $2|k$, we know that $K \subseteq \mathbf{F}$ and we conclude that $\mathbf{F}$ must contain a root ω of f. □

These results allows us to characterize the finite fields over which the combinatorially complete Hesse configuration is realizable, but there are many other cases to consider, each with its own question of realizability. The task of investigating the Hesse configuration's combinatorially degenerate flavors seems daunting at first glance. We carefully and systematically examine these possibilities with the intent of reducing the number of necessary considerations as painlessly as possible. We begin by addressing the more easily-digestible examples.

Lemma 3.13. *Any realization of the Hesse configuration on exactly one line that contains n distinct points is universally realizable if $1 \leq n \leq 9$.*

The remaining combinatorially degenerate Hesse configurations all contain at least two distinct lines, thereby complicating the question of realizability. The next-simplest class of these configurations, in which all points but one lie on a line, remains relatively easy to examine. First, we introduce a more general result pertaining to similar collections of points in the projective plane.

Proposition 3.14. *Let P be a subset of $n \geq 4$ distinct points in a projective plane. If at least three points are collinear in every subset of P containing four points, then at least $n - 1$ of the points in P are collinear.*

Proof. The result follows quickly for $n = 4$; we proceed by induction on n, assuming that the statement is true for $n - 1$. Consider now n distinct points $p_1, p_2, \ldots, p_n$, such that $p_1, p_2, \ldots, p_{n-2}$ are collinear. By the inductive hypothesis and without loss of generality, assume that p_{n-1} is not collinear with the first $n - 2$. Then there are three possibilities for the placement of p_n. First, suppose that p_n is not collinear with $p_1, p_2, \ldots, p_{n-2}$ but lies on a line containing p_{n-1} and p_i for $1 \leq i \leq n - 2$. Consider the four points p_n, p_{n-1}, p_j, and p_k, where $j \neq k$. No three of these points are collinear, contradicting the initial assumption. Next, suppose that p_n is not collinear with $p_1, p_2, \ldots, p_{n-2}$ and does not lie on a line containing p_{n-1} and p_i for $1 \leq i \leq n-2$. Consider the four points p_1, p_2, p_{n-1}, and p_n, no three of which are collinear. Thus, we reach a contradiction of the initial assumption. Finally, suppose that p_n is collinear with $p_1, p_2, \ldots, p_{n-2}$. The desired result follows immediately, and we conclude that at least $n - 1$ of the n points are collinear in the projective plane. □

This proposition allows us to consider the configurations in which all but one of their points lie on a line more generally. The following lemma extends this result, showing that many of the combinatorially degenerate varieties of the Hesse configuration cannot be realized over any field. It may be helpful to revisit Remarks 3.6 and 3.7 while examining the following proofs.

Lemma 3.15. *Let P be a set of $6 \leq n \leq 9$ distinct points in a projective plane such that exactly $n-1$ are collinear. Then P cannot be a realization of $\mathcal{H}$.*

Proof. Since every pair of points in $\mathcal{H}$ is joined by a line and each point is contained by exactly four lines, we know that P cannot be a realization of $\mathcal{H}$. □

A significant number of combinatorially degenerate cases still remain: we categorize these configurations with respect to the number of distinct points contained in their realizations from this point on, as opposed to the number of distinct lines. The two subsequent lemmas further reduce the number of cases necessary to consider using arguments of collinearity. Their results allow us to narrow our search and consider arrangements of at most six distinct points in the projective plane.

Lemma 3.16. *If exactly eight points in any realization of the Hesse configuration are distinct, then all eight are collinear.*

Proof. Without loss of generality due to the symmetry of $\mathcal{H}$, suppose that $A = B$. Immediately, we obtain the collinear triples (AEH), (AFG), and (ADI). Using Corollary 2.17, the triples (AEH) and (AEI) guarantee that A, E, H, and I are collinear. Applying Corollary 2.17 again, (GHI) requires A, E, G, H, and I to be collinear. Now (CEG) similarly places C on this line, (ADI) adds D, and (DEF) then guarantees full collinearity. □

Lemma 3.17. *If exactly seven points in any realization of the Hesse configuration are distinct, then all seven are collinear.*

Proof. We consider three cases without loss of generality, due to the symmetry of $\mathcal{H}$.

1. First, suppose that $A = B = C$. Immediately, we obtain the collinear triples (AEH), (AFG), (ADI), (AFI), (ADH), and (AEG). It follows from Corollary 2.17 that there are three lines ℓ_1, ℓ_2, and ℓ_3 such that $A, E, G, H \in \ell_1$, $A, F, G, I \in \ell_2$, and $A, D, H, I \in \ell_3$. Applying Corollary 2.17 again shows that $\ell_1 = \ell_2 = \ell_3$, thereby implying the full collinearity of $\mathcal{H}$.

2. Next, suppose that $A = B = D$. We obtain the collinear triples (AEF), (AEH), (AFG), and (ACH) from this equality. Corollary 2.17 requires three lines ℓ_1, ℓ_2, and ℓ_3 such that $A, E, F, H \in \ell_1$, $A, E, F, G \in \ell_2$, and $A, C, E, H \in \ell_3$. Applying Corollary 2.17 again guarantees that $\ell_1 = \ell_2 = \ell_3$, thereby implying the full collinearity of $\mathcal{H}$.

3. Finally, suppose that $A = B$ and $D = E$, both without loss of generality. The collinear triples (ADI), (ADH), (AFG), and (CDG) are induced immediately. Corollary 2.17 shows that there are three lines ℓ_1, ℓ_2, and ℓ_3 such that

$A, D, H, I \in \ell_1$, $C, D, E, G \in \ell_2$, and $A, F, G, H \in \ell_3$. As above, applying Corollary 2.17 again requires that $\ell_1 = \ell_2 = \ell_3$, thereby implying the full collinearity of the realization of $\mathcal{H}$.

With all possible cases considered, the given statement is confirmed. □

The remaining combinatorially degenerate realizations of the Hesse configuration can be sorted neatly into two categories: those which contain five distinct points where no three are collinear, and those which contain no such five points. The next lemma shows that configurations belonging to the former cannot be realized in any projective plane.

Lemma 3.18. *If some subset of a realization of $\mathcal{H}$ contains five distinct points, then three points in this subset induce a collinear triple.*

Proof. Let p_1, p_2, p_3, p_4, p_5 be distinct points in the projective plane, no three of which induce a collinear triple. For the sake of contradiction, we assume that these points form a subset of $\mathcal{H}$ – this proof attempts to 'construct' the Hesse configuration with these points. It follows from Remark 3.7 that ten distinct lines $\{e_i\}_{i=1}^{10}$ are determined by these points, each joining exactly two points of the subset. Without loss of generality due to the symmetry of $\mathcal{H}$, let $A = p_1$ and $B = p_2$, supposing that $A, B \in \ell_1$, $A, p_3 \in \ell_2$, $A, p_4 \in \ell_3$, and $p_3, p_4 \in \ell_4$. We know that $C \in \ell_1$ from the collinear triple (ABC) and claim that either $C = A$ or $C = B$.

To prove the claim, suppose that $C \neq A$ and $C \neq B$ for the sake of contradiction. By Lemma 3.6, we know that exactly four lines must pass through C. Three of these must be distinct, joining C with p_3, p_4, and p_5, respectively. These three lines must also be distinct from the aforementioned ten lines, as $C \neq A$ and $C \neq B$. Then thirteen distinct lines exist in the configuration, but $\mathcal{H}$ admits at most twelve such lines. We thus reach a contradiction and conclude either that $C = A$ or that $C = B$.

Now without loss of generality, let $D = p_3$. By similar logic as above, we have that either $G = A$ or $G = D$ to satisfy (ADG). We also know that without loss of generality one of F and H coincides with p_4. We know that $F \neq p_4$ since neither A nor D lies on the line containing B and p_4. Similarly, we know that $H \neq p_4$ since D does not lie on the line containing B and p_4. Thus, we reach a contradiction and conclude that p_1, p_2, p_3, p_4, p_5 cannot form a subset of the Hesse configuration. □

Now only one class of combinatorially degenerate realizations of the Hesse configuration remains to be examined: those containing four distinct points such that no three are collinear. The following series of lemmas allows us to show that these configurations cannot be realized over any field.

Lemma 3.19. *The Hesse configuration cannot be realized over exactly four distinct points such that no three induce a collinear triple.*

Proof. For the sake of contradiction, suppose that $\mathcal{H}$ is realized over a set of distinct points $P = \{p_1, p_2, p_3, p_4\}$. Without loss of generality, let $A = p_1$, $B = p_2$, and

$D = p_3$. Then to satisfy the collinear triples (ABC), (ADG), and (BDI), we require that C be equal to one of A and B, that G be equal to one of A and D, and that I be equal to one of B and D. Now we see that $E \neq p_4$ by (AEI), so p_4 is equal to at least one of F and H (the only remaining points in $\mathcal{P}$). If $p_4 = F$, then we violate (BFG). The collinear triple (CDH) is similarly violated if $p_4 = H$. We thus reach a contradiction, so $\mathcal{H}$ cannot be realized over P. □

We may combine the results of the two previous lemmas to write that if a subset $P = \{p_1, p_2, p_3, p_4\}$ of a realization of the Hesse configuration contains four distinct points such that no three induce a collinear triple, then there is a point $P \in \mathcal{P}$ distinct from P and that forms the collinear triple (Pp_ip_j) for some i, j. The lemmas below further extend this result.

Lemma 3.20. *The Hesse configuration cannot be realized over five distinct points where exactly three induce a collinear triple.*

Proof. For the sake of contradiction, suppose that $\mathcal{H}$ is realized over a set of distinct points $P = \{p_1, p_2, p_3, p_4, p_5\}$. Note that Remark 3.7 requires p_1 and p_2 to be contained by four distinct lines and the remaining points to be contained by three. Without loss of generality, let $A = p_1$, $B = p_2$. We know without loss of generality that one of D and G is equal to p_3, that one of E and I is equal to p_4, and that one of F and H is equal to p_5. The remaining points must either be equal to A or to their respective p_i. First, suppose that $p_3 = D$. Then (BDI) requires either that $I = B$ or $I = D$, neither of which is possible. Suppose now that $p_3 = G$. Then (BFG) requires that F be equal to either B or G, neither of which is possible. □

If $\mathcal{H}$ is realized over five distinct points $P = \{p_1, p_2, p_3, p_4, p_5\}$ that form only $(p_1p_2p_3)$ without loss of generality, then previous results allow us to conclude that there is a point $q \in \mathcal{P}$ distinct from P that induces another collinear triple.

Lemma 3.21. *The Hesse configuration cannot be realized over six distinct points that induce exactly two collinear triples.*

Proof. For the sake of contradiction, suppose that $\mathcal{H}$ is realized over a set of distinct points $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$. Without loss of generality, we consider two cases.

1. First, suppose that P forms the collinear triples $(p_1p_2p_3)$ and $(p_4p_5p_6)$. Note that Remark 3.7 requires that each point in P be contained by four distinct lines in this case. Without loss of generality, let $A = p_1$, $B = p_2$. We know without loss of generality that one of D and G is equal to p_3, that one of E and I is equal to p_4, and that one of F and H is equal to p_5. The remaining points must either be equal to A or to their respective p_i. First, suppose that $p_3 = D$. Then (BDI) requires either that $I = B$ or $I = D$, neither of which is possible. Suppose now that $p_3 = G$. Then (BFG) requires that F be equal to either B or G, neither of which is possible.

2. Next, suppose that $\mathcal{P}$ induces the collinear triples $(p_1p_2p_3)$ and $(p_1p_4p_5)$. Remark 3.7 requires that p_2, p_4, and p_6 be contained by four distinct lines, while the remaining points are contained by three. Without loss of generality, let $A = p_2$, $B = p_1$, and $D = p_4$; it follows that G is equal to one of A and D. Without loss of generality, one of E and I is equal to p_5, and one of F and H is equal to p_6. It follows by construction that $C = p_3$. Then since I is collinear with both A and B, the configuration guarantees that $I = p_5$. It follows from (CFI) that $F = p_6$. Then (AEI) requires that E be equal to one of A and I, both of which violate (DEF).

In either case we reach a contradiction, thus confirming the given statement. □

It follows that if a subset $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ of a realization of the Hesse configuration forms exactly one or two collinear triples, then there is a point $q \in \mathcal{P}$ distinct from P. We know, however, that any combinatorially degenerate Hesse configuration that contains seven distinct points that lie on more than one line cannot ever be realized. At this point, we have successfully described all combinatorially-possible Hesse configurations. This fact may be difficult to believe, but the symmetry of $\mathcal{H}$ allows us to generalize many of its combinatorial degeneracies; a close reading of the previous lemmas will affirm this fact. A natural question now emerges: what can be said about the Hesse configuration's realizability? This section's results have shown that many combinatorially-possible Hesse configurations are never realizable; the table below outlines those with realizability.

Combinatorial Description of Realization of $\mathcal{H}$	Realizability
All points collinear with any number of them distinct	Universal
Three distinct points with exactly two collinear	Universal
Four distinct points with exactly three collinear	Universal
Five distinct points with exactly four collinear	Universal
Combinatorially complete	All $\mathbf{F}$ containing $\sqrt{-3}$

Table 1. All combinatorial possibilities over which $\mathcal{H}$ may be realized.

It should be noted that $\mathcal{H}$ shares similar properties of realizability with another symmetric configuration, 8_3. In particular, both combinatorially complete configurations are only realizable over fields containing $\sqrt{-3}$, and the combinatorially degenerate flavors of these configurations are realizable either never or always. These proofs related to 8_3 have been omitted from this thesis for the sake of reducing redundancy, but follow nearly-identical logic as those presented throughout this section. A larger discussion of the connections via realizability of various symmetric configurations remains open, though we predict that there may be powerful underlying connections between the structures of configurations and their corresponding fields. In the next chapter, we examine at length various asymmetric configurations and discuss how to "encode" an algebraic constraint into a projective configuration.

CHAPTER 4. THERE AND BACK AGAIN

In the previous chapter, we investigated symmetric projective configurations due to their ease of understanding. Their symmetries allowed us to utilize many tools – transformations, coordinates, etc. – in examining their realizability. In the process, we showed that if the combinatorially complete Fano configuration is realized over a projectivized vector space, then the underlying field must have characteristic 2. Similarly, the combinatorially complete Hesse configuration must be realized over a field containing $\sqrt{-3}$. We can write these conditions algebraically, by $x + x = 0$ and $x^2 - x + 1 = 0$, respectively. A natural question arises from this observation: can we 'encode' any such algebraic condition into a projective configuration, and if they exist, what do these configurations look like? In this chapter, we will discuss a powerful construction that allows us to create such configurations and investigate their structures.

4.1. Geometric Operations in the Projective Plane

In Chapter 2, we mentioned that if Pappus's Theorem holds in a projective plane (and thereby Desargues's Theorem also holds), then that plane can be coordinatized over a field. This claim was bold to make without proof, although Marchisotto does so beautifully in her 2002 paper [11]. She approaches the proof axiomatically, by expanding Definition 2.1 with three additional assumptions. These axioms are:

4. Every line contains at least three points.

5. The three diagonal points of a complete quadrangle are not collinear.

6. A transformation that fixes three distinct, collinear points fixes the whole line.

Marchisotto adopts these axioms to narrow our study to a world in which Pappus's Theorem holds. As she proves in her paper, this world inherently connects the projective plane to an underlying algebraic field structure. In the process, Marchisotto also defines geometric operations in the projective plane that correspond to addition, subtraction, multiplication, and division in the algebraic sense. In this section, we outline these operations for the purposes of our thesis, then make use of them to 'encode' algebraic constraints into projective configurations. For each operation, we let ℓ be a line containing the points O, E, and M. The point O will serve as the additive identity, E will serve as the multiplicative identity, and M will serve as an ideal point, or "infinity." We define the four operations on $\ell \setminus \{M\}$ in order to yield a field-based coordinate system [11].

We first investigate the geometrically-defined additive operation. Let A, B be points on $\ell \setminus \{M\}$. Choose P to be a point not on ℓ, and Q to be a point on the line OP. Then define $R = QM \cap PA$ and $S = PM \cap QB$. Let $C = RS \cap \ell$; then we define $C := A + B$. From this construction, we ascertain that O corresponds to the additive identity element. Note also that E plays no role in this construction; we will show shortly that E corresponds to the multiplicative identity element [11]. A depiction of geometrically-defined addition can be found below in Figure 12.

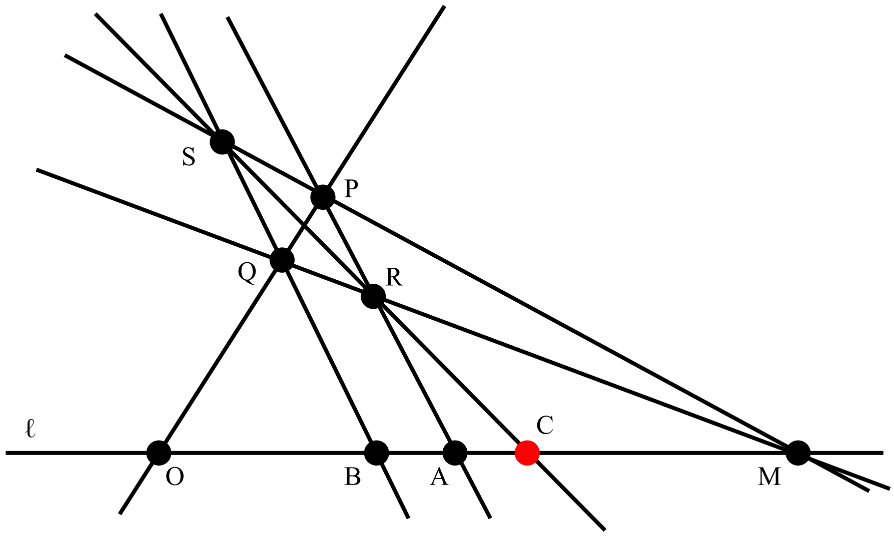

Figure 12. A depiction of $A + B = C$ using geometrically-defined addition.

We define subtraction by finding a point $-B \in \ell$ such that $-B + B = 0$, using addition as defined above. The subtraction of B from A then corresponds to the addition of A and $-B$. Expanding upon the construction of addition above, define $L = QM \cap PB$ and $K = OR \cap PM$. Let $-B = KL \cap \ell$; then repeating the process of geometrically-defined addition reveals that $-B + B = O$, as desired.

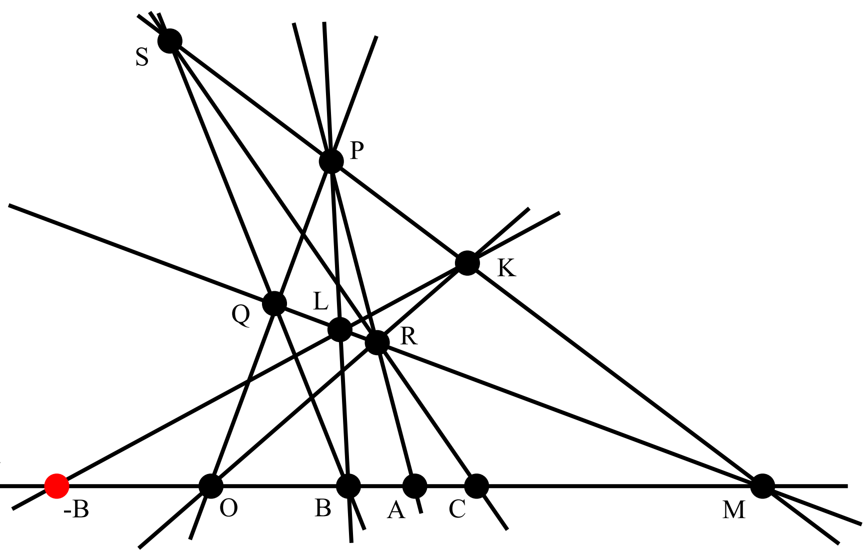

Figure 13. Using the equation $A + B = C$, we find the point $-B \in \ell$.

Next, we outline the process of obtaining geometrically-defined multiplication in the projective plane. Once again, let A, B be points on $\ell \setminus \{M\}$ and pick P to be a point not on ℓ. Then choose a point Q on the line EP. We define $R = AP \cap QM$

and $S = PO \cap QB$. Let $C = RS \cap \ell$; then $C := A \star B$. Unlike addition, we note that both O and E contribute to this construction, from which we can show that E corresponds to the multiplicative identity element. For both addition and multiplication, the proofs of closure and associativity fall directly from the axioms assumed by Marchisotto [7, 11].

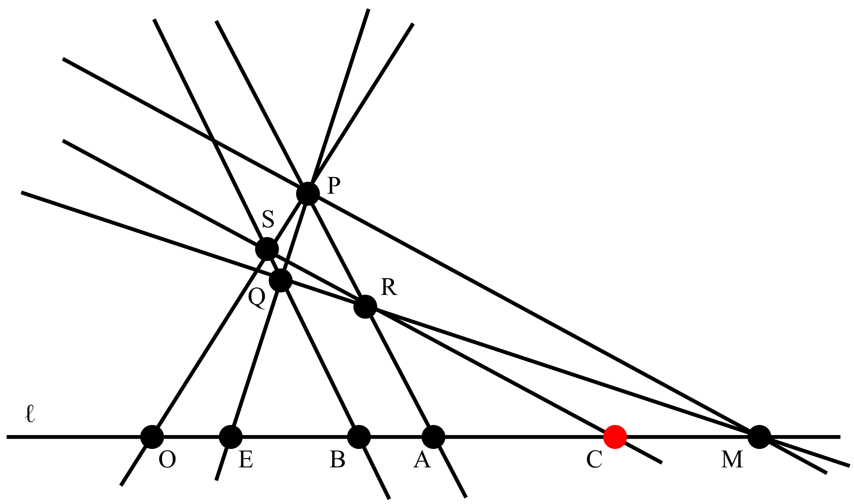

Figure 14. A depiction of $A \star B = C$ using geometrically-defined multiplication.

Geometrically-defined division is constructed in a similar manner to subtraction; we seek the multiplicative inverse of a given point on $\ell \setminus \{M, O\}$. As with multiplication, let B be a point on $\ell \setminus \{M, O\}$ and pick P to be a point not on ℓ. Then choose a point Q on the line EP. We seek the point $B^{-1} \in \ell \setminus \{M\}$ such that $B \star B^{-1} = E$. Define $L = QM \cap PB$ and $K = PO \cap EL$. Let $B^{-1} = QL \cap \ell$; then repeating the process of geometrically-defined multiplication shows that $B^{-1} \star B = E$. If we were to allow $B = M$, then $B^{-1} = O$. However, since we require $B \in \ell \setminus \{M, O\}$, division by zero can never occur. Thus, this geometrically-defined operation directly corresponds to division as we know it in the algebraic sense [11].

Geometrically-defined addition is commutative based upon its construction, and it follows from the assumed axioms that geometrically-defined multiplication is also commutative. One may also verify that geometrically-defined multiplication distributes over addition. A natural question is whether these operations rely on our choices of P and Q, for instance; Marchisotto shows that all four of these operations are well-defined. Proofs for all of these properties can be found in [11], though they remain omitted from this thesis for the sake of clarity. Thus, we conclude that these operations (in conjunction with the axioms that render them valid) identify a field structure with $\ell \setminus \{M\}$ and allow us to coordinatize the projective plane as $P\mathbf{F}^2$.

It is natural to wonder how we might extend the reach of these operations. Callie Garst [7] defined a geometrically-defined "square root" operation, allowing us to locate the point $X^{k/2^n}$ for any non-negative $X \in \ell \setminus \{M\}$ and $k, n \in \mathbb{Z}$. This

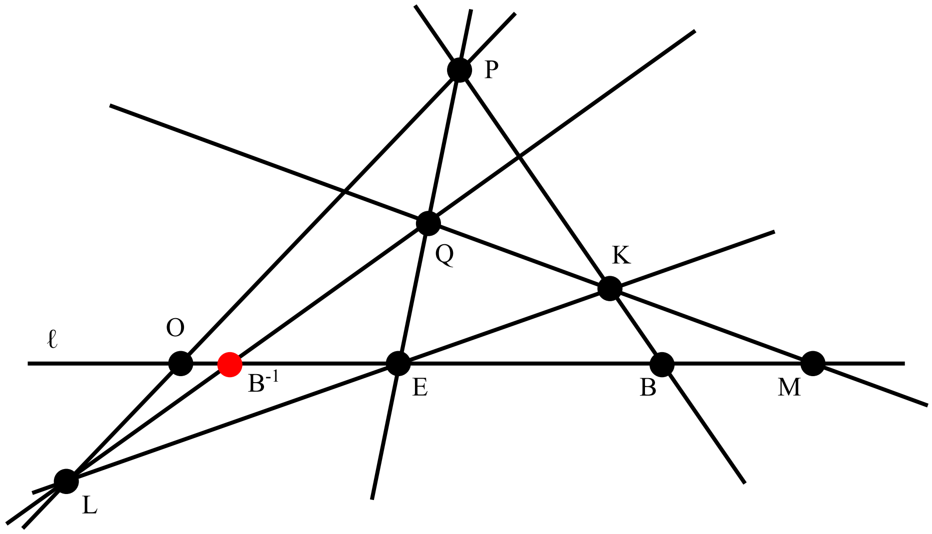

Figure 15. We find the point B^{-1} so that we may geometrically divide by B.

work allows us to define any polynomial equation over $\mathbf{F}$ of the following form using a geometric perspective in $P\mathbf{F}^2$:

$$c_1x^{e_1} + c_2x^{e_2} + \cdots + c_nx^{e_n} = 0,$$

where $c_1, c_2, \ldots, c_n \in \mathbf{F}$ and $e_1, e_2, \ldots, e_n \in \mathbb{Z}$ or $e_1, e_2, \ldots, e_n = k/2^n$ for $k, n \in \mathbb{Z}$. In the next section, we expand upon these operations and provide a method for creating configurations that correspond to a wide swath of polynomial equations over $\mathbf{F}$.

4.2. Encoding Algebraic Conditions

Our geometrically-defined operations allow us to visualize addition, subtraction, multiplication, and division in a field-based projective plane. In other words, we have a basis for performing arithmetic computations from a geometric perspective. We ask of you now, dear reader, to recall back to the days of elementary algebra. After learning the times tables, for example, many curricula transition to solving linear equations: having a grasp of the arithmetic operations allows us to explore more complex and beautiful mathematics. In this section, we hope to provide a similar expansion. Although it may not seem immediately intuitive, Marchisotto's geometrically-defined operations correspond to projective configurations in the plane. We define the configurations corresponding to addition and multiplication below.

Definition 4.1. The *addition configuration* is a configuration $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P} = \{O, A, B, C, P, Q, R, S, M\}$ and $\mathcal{L} = \{(OPQ)$, (BQS), (ARP), (CRS), (MRQ), (MPS), (OAB), (OAC), (OAM), (ABC), (ACM), $(BCM)\}$. A realization of $\mathcal{A}$ is *Marchisotto* if it renders M distinct from O, A, B and if it renders P not collinear with O, A, B, C, M.

Definition 4.2. The *multiplication configuration* is a configuration $\mathcal{M} = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P} = \{O, E, A, B, C, P, Q, R, S, M\}$ and $\mathcal{L} = \{(OPS)$, (EPQ), (ARP), (BQS),

(CRS), (MRQ), (OAB), (OEB), (OEA), (OAC), (OAM), (ABC), (ACM), and $(BCM)\}$. A realization of $\mathcal{M}$ is *Marchisotto* if it renders M distinct from O, A, B and if it renders P not collinear with O, E, A, B, C, M.

Proposition 4.3. *Let $\boldsymbol{V}$ be a three-dimensional vector space over $\mathbf{F}$, where the characteristic of $\mathbf{F}$ is not 2. Then $\mathcal{A}$ and $\mathcal{M}$ are Marchisotto realizable over $P\mathbf{F}^2$.*

In Chapter 1.3, we discovered that the realizability of many symmetric configurations hinges upon finding solutions to polynomial equations over $\mathbf{F}$. We now attempt the reverse: given a polynomial equation over $\mathbf{F}$, we will encode that equation into a projective configuration.

Definition 4.4. Let f be a polynomial equation over $\mathbf{F}$. The *encoding of f* is a configuration $\mathcal{E}$ such that a combinatorially complete realization of $\mathcal{E}$ exists over $P\mathbf{F}^2$ if and only if $\mathbf{F}$ contains a root to the equation.

To begin our discussion of encodings, we recall from Chapter 3 that a combinatorially complete realization of the Fano configuration requires that $P\mathbf{F}^2$ has characteristic 2. When considering the corresponding geometrically-defined operations of the equation $x + x = 0$, we note that $A = B$ and set $C = O$. This encoding is defined below.

Definition 4.5. The *Encoding of $x+x=0$* is a configuration $\mathcal{E}_1 = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P} = \{O, A, P, Q, R, S, M\}$ and $\mathcal{L} = \{(OPQ)$, (AQS), (ARP), (ORS), (MRQ), (MPS), $(OAM)\}$. A realization of $\mathcal{E}_1$ is *Marchisotto* if it renders M distinct from O, A and if it renders P not collinear with O, A, M.

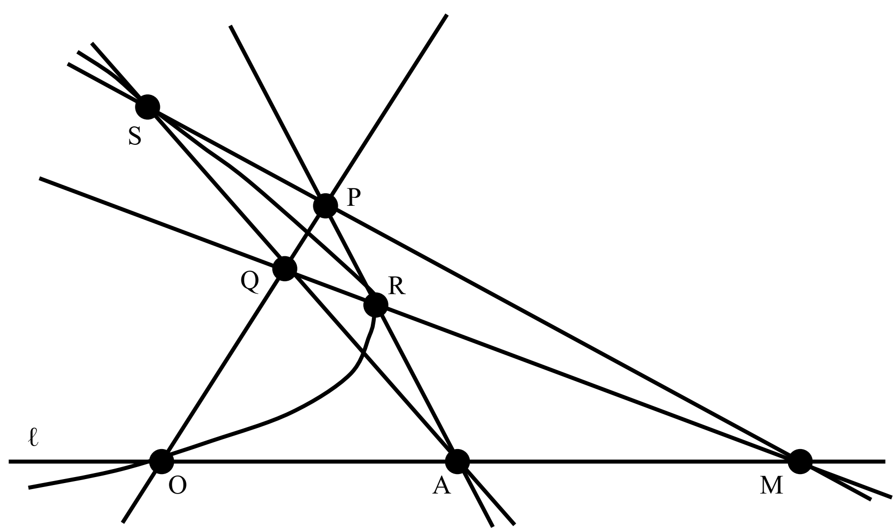

Figure 16. A depiction of the Encoding of $x + x = 0$.

The astute reader may note that any combinatorially complete realization of $\mathcal{E}_1$ violates Marchisotto's assumption of the axiom 5 above. Here, we emphasize the crucial distinction between geometrically-defined addition and the configuration based upon its construction. We note that $\mathcal{E}_1$ is a symmetric configuration by construction. In particular, we see that any combinatorially complete realization of $\mathcal{E}_1$ consists of seven points, each of which lies on exactly three lines of $\mathcal{L}$. It follows that $\mathcal{E}_1$ is isomorphic to the Fano configuration $\mathcal{F}$. We conclude that any combinatorially complete realization of $\mathcal{E}_1$ in $P\mathbf{F}^2$ implies that $\mathbf{F}$ has characteristic 2. It can be shown that any combinatorially degenerate realization of $\mathcal{E}_1$ is universally realizable. Not all encodings yield symmetric configurations that ameliorate our studies, though. We investigate one such encoding below.

Recall that a combinatorially complete realization of the Hesse configuration exists in $P\mathbf{F}^2$ if and only if there is an $x \in \mathbf{F}$ such that $x^2 - x + 1 = 0$. To encode this condition in a configuration, we perform three geometrically-defined operations in the projective plane. We first use geometrically-defined multiplication to perform the computation $A \star A = B$. Next, we use geometrically-defined subtraction to yield $B - A = C$. We apply geometrically-defined addition once more to calculate $E + C = D$ and set $D = O$. For both addition and multiplication, we recall that the choice of P is arbitrary [7, 11]. For the sake of convenience, then, we let P be the same for both operations and choose Q_+, $Q_\star$ to apply for every use of their respective operations.

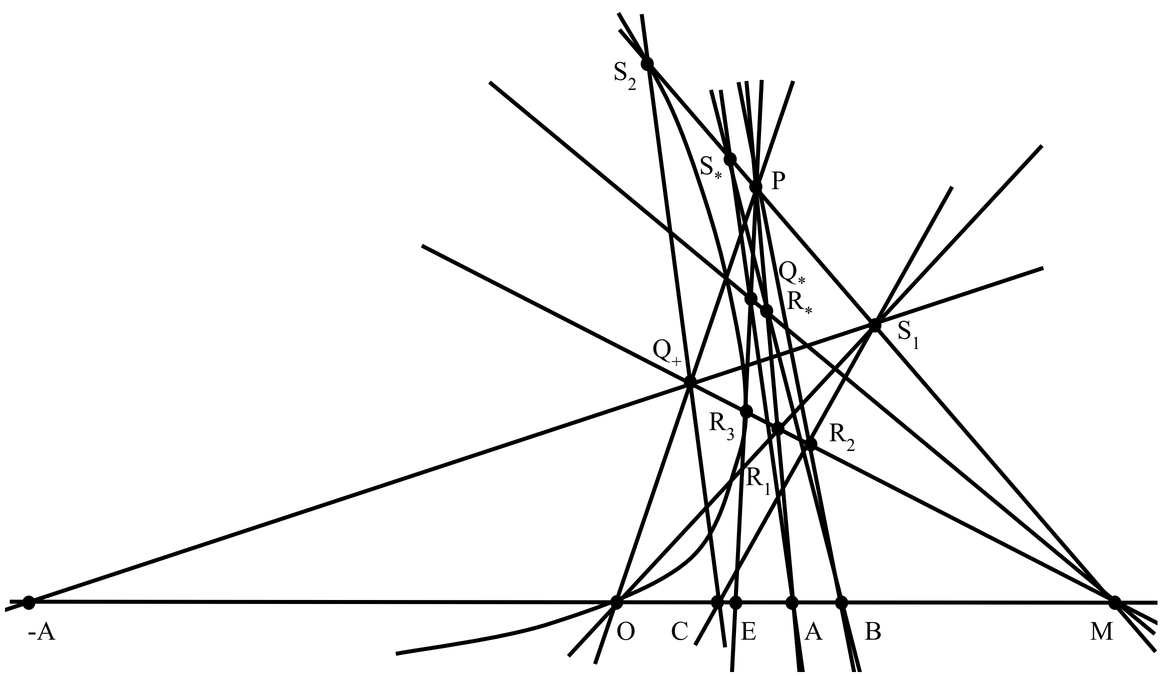

Figure 17. A depiction of the Encoding of $x^2 - x + 1 = 0$.

Definition 4.6. The *Encoding of* $x^2 - x + 1 = 0$ is a configuration $\mathcal{E}_2 = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P} = \{O, E, A, -A, B, C, P, Q_\star, R_\star, S_\star, Q_+, R_1, R_2, R_3, S_1, S_2, M\}$ and $\mathcal{L} = \{(OQ_+S_\star)$, (OQ_+P), $(OS_\star P)$, (OR_1S_1), $(EQ_\star P)$, $(EQ_\star R_3)$, (ER_3P), $(AR_\star P)$, $(AR_\star R_1)$, (AR_1P), $(AQ_\star S_\star)$, $(-AQ_+S_1)$, (BR_2P), $(BR_\star S_2)$, (CQ_+S_2), (CR_2S_1), $(Q_\star R_\star M)$, (PS_1S_2),

(PS_1M), (PS_2M), (OR_3S_2), (OEA), (OEB), $(-AOE)$, $(-AEA)$, (OEA), (ABC), (ABM), $(ACM)\}$. A realization of $\mathcal{E}_2$ is *Marchisotto* if it renders M distinct from $O, E, A, -A, B, C$ and if it renders P not collinear with $O, E, A, -A, B, C, M$.

Theorem 4.7. *A combinatorially complete and Marchisotto realization of $\mathcal{E}_2$ in $P\mathbf{F}^2$ exists if and only if some $x \in \mathbf{F}$ satisfies $x^2 - x + 1 = 0$.*

Proof. Suppose first that a combinatorially complete realization of $\mathcal{E}_2$ in $P\mathbf{F}^2$ exists. Then using the geometrically-defined operations, we see that $A \star A - A + E = O$. Now suppose that some $x \in \mathbf{F}$ satisfies $x^2 - x + 1 = 0$. Then we choose x to correspond with $A \in \mathcal{P}$ and can construct $\mathcal{E}_2$ as defined. □

Unlike the Fano configuration and $\mathcal{E}_1$, however, we note now that $\mathcal{E}_2$ is not symmetric and therefore is not isomorphic to the Hesse configuration. We also note that there is a degree of flexibility with our construction; since our choices of $P, Q_+, Q_\star$ allow for four degrees of freedom, we see that some choices "reduce" the resulting configuration. For example, if we pick $Q_+, Q_\star$ to form the collinear triple $(Q_+Q_\star M)$, then $R_1 = R_\star$. Regardless of these reductions, though, $\mathcal{E}_2$ remains not isomorphic to $\mathcal{H}$. We hope that future work will further investigate these reductions and provide a more rigorous understanding of their flexibility. Nevertheless, we have provided a preliminary method for encoding any polynomial equation over $\mathbf{F}$ of the following form into a projective configuration in $P\mathbf{F}^2$:

$$c_1x^{e_1} + c_2x^{e_2} + \cdots + c_nx^{e_n} = 0,$$

where $c_1, c_2, \ldots, c_n \in \mathbf{F}$ and $e_1, e_2, \ldots, e_n \in \mathbb{Z}$ or $e_1, e_2, \ldots, e_n = k/2^n$ for integers k, n. The results of this chapter hinged on geometrically-motivated operations and constructions. In the next chapter, we will introduce topological intuition to our study as we define a family of incidence theorems that originate from projective configurations, then provide an investigation of the family's realizability.

CHAPTER 5. A WHOLE NEW WORLD

Until now, this thesis has investigated two-dimensional projective configurations made up of one-dimensional lines and zero-dimensional points. The requirement that our configurations be embeddings in the projective plane arises primarily from our focus on collinear triples: three points required to lie on the same line. We note that by definition, a collinear triple defines a degenerate triangle. A natural and simple question arises: "Can we learn anything about configurations in the projective plane by requiring these triangles to be non-degenerate?" By allowing our one-dimensional objects to acquire a second dimension, the resulting configurations they determine may extend into a third dimension. This new understanding of projective configurations allows us to infuse our study of incidence theorems with some topological intuition. In this chapter, we will outline a family of incidence theorems that arise from this perspective and provide two proofs of their realizability.

5.1. Triangulations and Incidence

When first introducting the notion of projective configurations in Chapter 2, we provided Lemma 2.16 as a motivating example for a fully collinear and universally realizable configuration. In particular, we showed that for a set of four points, three collinear triples guarantee full collinearity of the whole set. This result also assisted in many of the proofs in Chapter 3. As mentioned previously, it can often seem difficult to visualize the interplay of many collinear triples abstractly. Thus, picturing these triples as formerly non-degenerate triangles provides a greater ease of understanding. In this new context, we may rephrase Lemma 2.16 as follows: if three faces of a tetrahedron degenerate to lines, then so too does the remaining face. Hence, the lemma earns its name.

In this chapter, we once again use Lemma 2.16 to inform a much broader study. Before presenting our main theorem and its results, though, we provide some preliminary definitions and background knowledge in elementary topology. Just as a curve generalizes the concept of a line, so too does a *surface* generalize the concept of a plane. Lines and planes must exhibit "straightness," while curves and surfaces are not subject to this requirement. A surface embedded in three-dimensional space is called *closed* if it is the boundary of a solid; otherwise, it is *open*. More technically, a closed surface is compact and without boundary. Topologists distinguish between surfaces by investigating properties called invariants. For example, connectedness is a topological invariant: if two surfaces differ in their number of connected components, then they are not homeomorphic (read: the "same").

In the 1700s, Leonhard Euler attempted to categorize surfaces by describing their structure regardless of bending or flexibility. Known as Euler characteristic, this metric was proven after Euler's death to be a topological invariant and is one of the most well-known elements of mathematics to date. Euler characteristic relies on shockingly simple machinery to understand: counting the faces, edges, and vertices present in a simplicial triangulation of a surface. The main theorem presented in this chapter also hinges upon simplicial triangulations.

Definition 5.1. A *(two-dimensional) simplicial triangulation* is a triple $\mathcal{T} = (V, E, T)$ that satisfies the following:

1. V is a finite set of vertices.
2. E is a set of edges determined by unordered pairs $\{A, B\}$ where $A, B \in V$.
3. T is a nonempty set of triangles determined by unordered triples of vertices $\{A, B, C\}$ where $A, B, C \in V$.
4. If $\triangle \in T$ and $\{A, B\} \subseteq \triangle$, then $\{A, B\} \in E$.

As defined, triangulations can be drawn using pictures that correspond to geometric objects. In Figure 18 below the triangulations correspond to the disk D^2. We say that a simplicial triangulation is *happy* if it satisfies the property that for every $\triangle \in T$, there exists a finite, nonempty subcomplex $S_\triangle = (V_\triangle, E_\triangle, T_\triangle)$ of $\mathcal{T}$ such that each edge in $E_\triangle$ is contained in exactly two elements of $T_\triangle$ and $\triangle \in T_\triangle$. We note that every simplicial triangulation of a closed surface is happy.

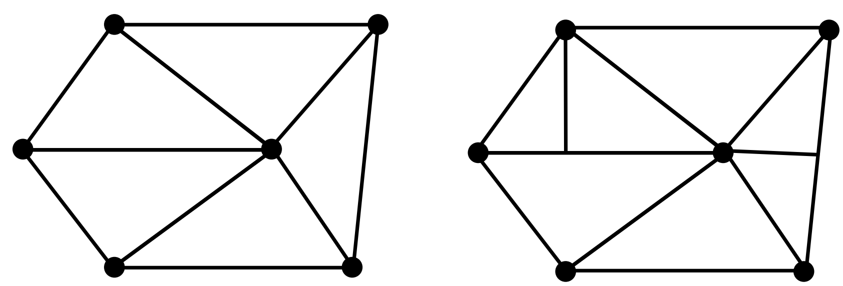

Figure 18. The left triangulation satisfies Definition 5.1; the right does not. Neither triangulation is happy, since only one triangle contains each boundary edge.

There are an infinite number of valid simplicial triangulations for any object, since each triangle can be subdivided into at least three more triangles by placing points and line segments in its interior. See [1, 2, 3] for more details. We note that the smallest triangulation of a closed surface consists of four vertices and four triangles that determine a tetrahedron.

Definition 5.2. Let $\mathcal{T} = (V, E, T)$ be a happy simplicial triangulation. We say that $\mathcal{T}$ contains a *subdivision* if there are vertices x, y, and z in V and edges $\{x, y\}$,$\{y, z\}$, and $\{x, z\}$ in E, but $\{x, y, z\}$ is not an element of T.

Definition 5.3. Let $\mathcal{T} = (V, E, T)$ be a simplicial triangulation. The *corresponding configuration to* $\mathcal{T}$ is (V, T).

For example, consider a planar tetrahedron, which is a simplicial triangulation of the disk D^2. Let its four vertices be denoted A, B, C, D. Then $E = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$. Similarly, $T = \{\{ABC\}, \{ABD\}, \{ACD\}, \{BCD\}\}$. The corresponding configuration to this triangulation is (V, T), which informs Lemma 2.16. The difference between a simplicial triangulation and its corresponding configuration is subtle and becomes visible when realizations are considered. In particular, a simplicial triangulation does not require much structure from the triples in T, while any realization of its corresponding configuration forces the triples to be collinear.

Definition 5.4. Let $\mathcal{T} = (V, E, T)$ be a happy simplicial triangulation. A realization of (V, T) is *orientable* if it is possible to assign orientations to each triangle in T such that for every edge $e \in E$, the two triangles containing e have opposite orientations.

Theorem 5.5. *Let $\mathcal{T} = (V, E, T)$ be a happy simplicial triangulation. Suppose that $\triangle_n \in T$ and consider the configuration $\mathcal{T}' = (V, T \setminus \{\triangle_n\})$. In any orientable realization of this configuration, the vertices of $\triangle_n$ are collinear.*

We now provide two proofs of this theorem. The first proof, while more accessible, requires more complicated machinery and thus restricts the number of projective planes over which $\mathcal{T}'$ may be realized. Just as with the Sylvester-Gallai Theorem presented in the first chapter, our second proof uses arguments of collinearity (and is thereby slightly more involved) to apply much more generally.

Field-Based Proof of Theorem 5.5. Let $\mathbf{F}$ be a field with characteristic not equal to 2, and let $\mathcal{T}'$ be realized over $P\mathbf{F}^2$. For the sake of contradiction, suppose that the vertices of $\triangle_n$ are not collinear in the realization of $\mathcal{T}'$. We assume that there is a line $\ell \in P\mathbf{F}^2$ that contains no points of the realization; the proof will only apply in this case. We declare this ℓ to be the ideal line and note that $P\mathbf{F}^2 \setminus \{\ell\}$ is an affine plane. We will work in $P\mathbf{F}^2 \setminus \{\ell\}$ for the remainder of this proof so that area calculations bear some meaning. We assign each line segment in the realization an (arbitrary) orientation such that we travel in the direction from v_i to v_j for $i < j$. We attribute the value $x_i y_j - x_j y_i$ to every such edge e_{ij}, where x_i, x_j, y_i, y_j correspond to the respective first and second affine coordinates of v_i and v_j.

We claim that three vertices v_i, v_j, v_k are collinear if and only if $e_{ij} - e_{ik} + e_{jk} = 0$. Recall that the area of a triangle in the plane is determined by its vertices:

$$\text{if } T = v_i v_j v_k, \text{then Area}(T) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{vmatrix}.$$

Suppose that v_i, v_j, v_k are collinear, so $\text{Area}(T) = 0$. Then the resulting determinant $x_i y_j - x_j y_i - x_i y_k + x_k y_i + x_j y_k - x_k y_j = e_{ij} - e_{ik} + e_{jk} = 0$. Now suppose that $e_{ij} - e_{ik} + e_{jk} = 0$. Then $x_i y_j - x_j y_i - x_i y_k + x_k y_i + x_j y_k - x_k y_j = 0$, so $\text{Area}(T) = 0$. Thus, v_i, v_j, v_k are collinear.

Recall that the realization of $\mathcal{T}'$ is orientable, and assign an orientation to each triangle in the realization; without loss of generality, choose "counterclockwise," as depicted in Figure 19. Notice that we may calculate the total area of two adjacent and oriented triangles by adding their respective areas. Let $T_1 = v_1v_3v_2$ and $T_2 = v_1v_2v_4$ (denoted such that the vertices occur in the order of the orientation of the triangles). Then $2\text{Area}(T_1) = e_{13} - e_{23} - e_{12}$ and $2\text{Area}(T_2) = e_{12} + e_{24} - e_{14}$. It follows that $2\text{Area}(T_1 + T_2) = e_{13} - e_{23} - e_{14} + e_{24}$. Notice that the e_{12} terms vanish from this sum. In other words, terms corresponding to shared edges of adjacent triangles do not appear in calculations of total area.

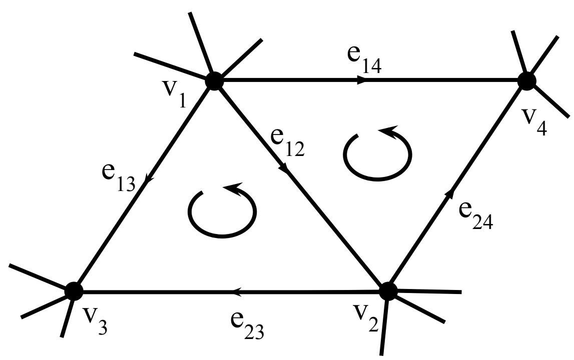

Figure 19. Orienting the triangles T_1, T_2 and their respective edges.

We now compute the area of the realization of $\mathcal{T}'$ by adding the areas of its component triangles. Suppose that $\triangle_n = \{v_1, v_3, v_2\}$ (such that the vertices occur in the order of the orientation of the triangle). Recall that $\text{Area}(\triangle_i) = 0$ for $1 \leq i \leq n-1$ since the vertices of these triangles are collinear upon realization. Then:

$$\begin{aligned}\text{Total Area} &= \text{Area}(\triangle_1) + \text{Area}(\triangle_2) + \cdots + \text{Area}(\triangle_n)\\ &= 0 + 0 + \cdots + 0 + \text{Area}(\triangle_n).\end{aligned}$$

Now note that since $\mathcal{T}$ is a happy simplicial triangulation, exactly one other triangle in T contains each of the edges e_{12}, e_{23}, e_{13}, respectively. Then we have:

$$\begin{aligned}\text{Area}(\triangle_n) &= \text{Area}(\triangle_1) + \text{Area}(\triangle_2) + \cdots + \text{Area}(\triangle_n)\\ 2\text{Area}(\triangle_n) &= (e_{13} - e_{23} - e_{12}) + \cdots + e_{23} + \cdots + e_{12} + \cdots - e_{13}\\ &= (e_{13} - e_{13}) + \cdots + (e_{23} - e_{23}) + \cdots + (e_{12} - e_{12})\\ &= 0\end{aligned}$$

Then $\text{Area}(\triangle_n) = 0$ and we conclude that v_1, v_2, v_3 are collinear. □

Though rather effective, this proof makes use of machinery that restricts the number of projective planes over which $\mathcal{T}'$ may be realized. Since we rely on coordinates, this proof only applies to projective planes defined over fields. Further, since the area function utilizes the fraction 1/2, such a field cannot have characteristic equal to 2. For instance, this proof does not apply to the Fano plane. Finally, if the realization of the corresponding configuration to $\mathcal{T}'$ is “big enough” that no such $\ell \in P\mathbf{F}^2$ exists, this proof does not apply. To rectify this issue, we provide a more general, less restrictive proof for the theorem below.

Collinearity-Based Proof of Theorem 5.5. Let $\mathcal{T}'$ be realized in some projective plane. Let $\triangle_n = \{v_1, v_2, v_3\}$ and for the sake of contradiction, suppose that v_1, v_2, v_3 are not collinear in the realization. Then v_1, v_2, v_3 form a non-degenerate triangle in the projective plane. We will proceed by induction on k, the number of vertices in V. Recall that the the fewest number of vertices in a happy triangulation is four – they make four triangles that determine a tetrahedron – so we suppose first that $k = 4$. In this case, Lemma 2.16 yields the desired contradiction. For induction, assume that the statement holds for all $k \leq n$ and suppose now that $k = n + 1$. We consider the collection of vertices that share an edge with v_1 and label them without loss of generality as $p_1, \ldots, p_m$ in cyclic order, as shown below in Figure 20.

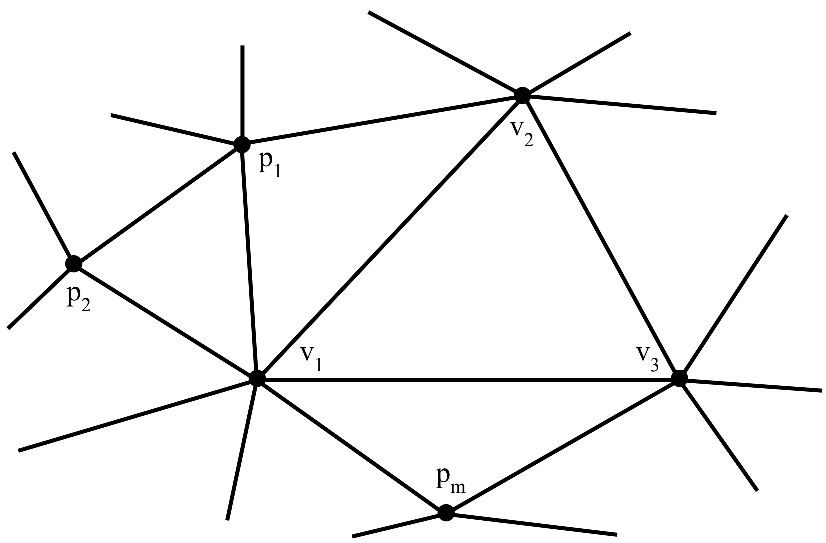

Figure 20. A local depiction of $\mathcal{T}'$ with $p_1, \ldots, p_m$ denoted about v_1, v_2, and v_3.

First, suppose that v_1 is distinct from each p_i and that each p_i is distinct from p_{i+1}. Since $\{v_1, v_2, p_1\} \in T$, it follows that p_1 must lay on the line v_1v_2. Similar logic necessitates that each p_i lay on the line v_1v_2, whence we consider the triple $\{v_1, v_3, p_n\} \in T$. Then v_3 must lay on the line v_1v_2, and so the three points are collinear in the realization of $\mathcal{T}'$.

Next, suppose that v_1 is not necessarily distinct from each p_i and that each p_i is not necessarily distinct from p_{i+1} in the realization. In other words, the realization of $\mathcal{T}'$ maps an edge in E to a point in the projective plane. Suppose that this edge is not part of a subdivision. Then this realization is equivalent to that of a simplicial triangulation with fewer vertices. Namely, this triangulation does not contain the contracted edge, nor does it contain the two triangles adjacent to the contracted edge, nor does it contain one of the vertices that determines the contracted edge. Abrams and Pommersheim illustrate this reduction well in [1]. In other words, this triangulation contains n vertices and so the induction hypothesis states that v_1, v_2, and v_3 are collinear.

Suppose now that the edge is part of a subdivision. Then this realization is equivalent to that of a simplicial triangulation with fewer vertices. If the subdivision occurs on the exterior of the triangle determined by v_1, v_2, and v_3, then the equivalent simplicial triangulation does not contain any of the triangles within the subdivision, as shown in [1]. It follows that this triangulation does not contain the omitted triangles' interior vertices. Then this triangulation consists of at most n vertices, and so the induction hypothesis states that v_1, v_2, and v_3 are collinear. If the triangle determined by v_1, v_2, and v_3 lays on the interior of the subdivision, then the equivalent simplicial triangulation contains at least one fewer triangle than the original, does not contain the contracted edge, and does not contain one of the vertices that determines the contracted edge. Thus, this triangulation consists of at most n vertices and the induction hypothesis states that v_1, v_2, and v_3 are collinear. □

This proof hinges only upon arguments of collinearity and thus allows the theorem to apply to a much broader swath of projective planes; in fact, this proof renders the theorem valid for every projective plane satisfying Definition 2.1. Thus, while more tedious and perhaps less visualizable, our collinearity-based proof acts as the Coxeter to our field-based proof's Kelly.

5.2. Moving Forward

Dear reader, we've come a long way over the course of the last five chapters. After building intuition about both axiomatic and field-based projective planes, we discussed various projective transformations belonging to $PGL_3(\mathbf{F})$. We then defined the various flavors of projective configurations and launched into an investigation of certain configurations and their realizability. In particular, we examined the Fano configuration and the Hesse configuration at length. Our results from Chapter 3 prove that these configurations can be realized over $P\mathbf{F}^2$ in a combinatorially complete fashion if and only if some $x \in \mathbf{F}$ satisfies either $x + x = 0$ or $x^2 - x + 1 = 0$, respectively. We expanded upon these conclusions in Chapter 4, where, using Marchisotto's geometrically-defined operations, we outlined how to encode into a configuration any polynomial equation over $\mathbf{F}$ of the form

$$c_1x^{e_1} + c_2x^{e_2} + \cdots + c_nx^{e_n} = 0,$$

where $c_1, c_2, \ldots, c_n \in \mathbf{F}$ and $e_1, e_2, \ldots, e_n \in \mathbb{Z}$ or $e_1, e_2, \ldots, e_n = k/2^n$ $k, n \in \mathbb{Z}$. In Chapter 5, we went down a different path and investigated how topological intuition influences our studies of configurations and realizability. Specifically, we defined happy simplicial triangulations and provided two proofs to a novel theorem in the process. Now, we look forward to where future researchers may pick up this labor.

Many natural questions emerge from the work of this thesis. The first chapter may inspire a budding mathematician to work (in the fashion of my thesis advisor) on area relations and equidissections. From the second chapter, one may wonder if a group of projective transformations exists over skew fields (or rings!) and if so, how to describe such an object. The third chapter bears the question of whether or not all combinatorially degenerate realizations of projective configurations hold over every projective plane. One may expand upon the fourth chapter and outline a technique for encoding *any* polynomial equation over **F** into a projective configuration. Finally, I hope that the fifth chapter compels future researches to infuse machinery from other disciplines into their studies. In addition to these listed above, Garst's thesis poses many open questions at its completion [7].

This thesis and its contents matter to me because I chose to make them matter; without passion, our work has no meaning. Thank you, dear reader, for allowing me to share my passions with you. If at the end of the day you do not understand the projective plane, configurations, or realizability, I hope that you have gained a greater appreciation for the importance of points and lines. The simple things do indeed have beautiful and nuanced complexity. Rest now, for the sun is setting on Campestria.

ite domum saturae, venit Hesperus, ite capellae.

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