CHAOS AND THE DYNAMICS OF QUADRATIC MAPPINGS

An honors thesis for the department of mathematics
at
Washington and Lee University

by

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Preface

The immediate aim of this paper is to systematically investigate dynamical systems governed by a certain class of difference equations. The results obtained do not, however, constitute an isolated piece in a sprawling typology of dynamical systems. Instead, the phenomena encountered in our setting are often, with appropriate modifications, characteristic of all dynamical systems. Exploring these analogies is a secondary aim of this paper.

The type of dynamical system that forms the basis of our inquiry is without question the best "window" upon dynamical phenomena in general. The study of dynamical systems involving, say, differential equations requires a higher degree of mathematical sophistication, and is at the same time more cluttered. Certain phenomena present in the simplest (one-dimensional) systems involving difference equations have no parallels in continuous dynamical systems involving less than three coupled differential equations.[1,2]

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[1]May, 466-7.

[2]Holden, New Scientist, 13.

Part I: Dynamics

1. Dynamics and Intuition

> The elegant body of mathematical theory pertaining to linear systems...and its successful application to many fundamentally linear problems in the physical sciences, tends to dominate even moderately advanced University courses in mathematics and theoretical physics. The mathematical intuition so developed ill equips the student to confront the bizarre behavior exhibited by the simplest non-linear systems, such as $X_{n+1}=aX_n(1-X_n)$. Yet such non-linear systems are surely the rule, and not the exception, outside the physical sciences.[1]

Until the emergence of chaos theory in the last twenty years there was a tendency to ignore non-linear dynamical models. Scientists in all fields tended to "habitually set the hard non-linear systems aside."[2] The spontaneous result of this bias was a blindness to complexity. The linear mathematics that they refused to relinquish left no room for it.

In the absence of non-linear elements, the dynamical models of scientists inevitably prescribed simple equilibrium as the limiting behavior. Animal populations were to methodically approach final equilibrium, the prevalence of diseases was to

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[1] May, 467.

[2] Gleick, 305-6.

stabilize, and so forth. In every science the script was the same. Erratic populations and recurrent epidemics were met with the intellectual equivalent of a shrug, and chalked up to "a multitude of independent components or...to random external influences."[3] Only complexity could breed complexity it was reasoned.

There was a widespread ignorance of the capacity of simple non-linear systems to generate this complex, even startling behavior. Linear models were so entrenched that applied scientists mistook linear mathematics for all of mathematics. Non-linear dynamics was implicitly assumed to be a mere cousin of linear dynamics, albeit a computationally perverse one.[4]

Scientists were completely oblivious to the radical differences between linear and non-linear dynamical systems. The tool that would rescue science from its bewilderment in the face of complexity lay in a state of disuse, and scientific inertia threatened to keep it there permanently. Its recognition would require a pioneer willing to see complexity stemming from simplicity.

The breakthroughs eventually came. Lorenz (1961) in meteorology, Robert May (1971) in population biology[5], and others finally set off the "chaotic revolution" after years of darkness. At an ever accelerating speed, researchers began to reexamine data that had in the past been dismissed "by saying there's noise, or...

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[3]Gleick, 303.

[4]Gleick, 303-17.

[5]Gleick, 11-23 and 69-77.

that the experiment is bad[6]," as possible manifestations of non-linear dynamics. Today, the inclination to ignore the possibility of patterns of "limiting behavior" more complex than simple equilibrium or periodicity has virtually disappeared among scientists. Unfortunately, this paradigm shift hasn't reached much beyond the fraternity of practicing scientists. As Robert May points out, even today's college students are generally unaware of the beauty and richness of non-linear dynamics.

It is hoped that, in part, this paper will reveal the dramatic contrast between linear and non-linear dynamics. To this end, a fair amount of space is devoted to examining linear first order difference equations (Part II) before entering into the treatment of non-linear first order difference equations (Part III). The utter simplicity of the linear case will serve as a springboard for appreciating the menagerie of periodic, nearly periodic, and chaotic (seemingly random) behavior found in the simple non-linear system.

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[6]James Yorke as quoted in Gleick, 68.

2. The Terminology of Dynamics

Dynamics is the study of the evolution of systems over time. The "system" under consideration can be as intricate as the case of a multi-step chemical process with numerous reagents, or as simple as the population of a single animal species.

A dynamical system is completely characterized by two things. The first is a domain, a phase space, which encompasses all the possible states of the system. The state of a chemical reaction could, for instance, be represented by a vector of n real positive variables capturing the concentrations of the n chemicals involved. The phase space would then be a "quadrant" of $\mathbf{R}^n$. The second half of a dynamical system is the rule(s) of motion, or the "evolution equation(s)." These rules are functions that guide the changes in state of the system over time.[1] While these rules may be differential equations instantaneously relating position in phase space to velocity in phase space, they may also be more manageable difference equations. It is the latter that will concern us in Parts II and III.

Moreover, our focus will be on a very restricted class of difference equations in a one-dimensional setting. Our phase space will be **R**, the real line, and our "rules of motion" will be difference equations of the form $x_{n+1}=f(x_n)$ where f is an analytic

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[1]Percival, 1.

function whose range is contained in its domain. (That the state of the system at time n+1 depends solely upon the state of the system at time n is a first order condition.[2]) Higher order systems[3], or multi-dimensional systems are of interest in their own right but are not discussed here.

Our concern, then, will be with the behavior of the sequence of iterates $(x_0, f(x_0), f(f(x_0)), f(f(f(x_0))),\ldots)$ [4] for all possible seed values x_0, i.e., for all of **R**. We will seek to identify regions of **R** called basins of attraction in which every seed value x_0 leads to a common attractor as n approaches infinity, and furthermore to characterize these attractors.[5]

These attractors range from simple equilibrium points to limit cycles of period p to the vastly more complex strange attractors.[6] Where the attractor is strange the behavior of the iterates of x_0 under f is said to be chaotic as they appear to be randomly generated. In reality there is an order underlying this seeming randomness--the strange attractor has a fractal[7] geometric form, and this form merely escapes the conventionally trained eye.[8]

By way of example, consider the real line **R** (x is in radians)

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[2]Goldberg, 54.

[3]One example of a 2nd-order system would be the Fibonnaci sequence $X_{n+2}=f(X_{n+1},X_n)$.

[4]Hereafter we will denote $f(f(f(x)))$ as $f^3(x)$ and so on.

[5]Devaney, 2-3.

[6]Moon, 21-23.

[7]See Appendix A.

[8]Crutchfield, 51.

under the "motion equation" $x_{n+1}=\cos(x_n)$. In this dynamical system, no matter what x_0 in **R** is used as a seed value, the iterates of x will converge to the single solution of $x=f(x)=\cos(x)$. The one basin of attraction is all of **R**, and the attractor is the single fixed point $x' \approx .739085133215$. This fixed point x' is referred to as attractive since there is an interval (here: **R**) around x' in which all points converge to x' under iteration. In other dynamical systems we find repellant fixed points which repel all nearby points. The slightest perturbation from a repellant equilibrium will drive the system further from equilibrium. The former are called stable equilibria, and the latter unstable.[9]

Dynamical systems with attractors are said to be dissipative. They are the mathematical analog of physical systems that have an internal friction which ensures that arbitrary volumes in phase space contract with the passage of time as transients decay.[10] All motion that persists in the long run "in a n-dimensional dissipative system must be on a structure that has a dimension less than n: this structure is an attractor and occupies a subspace X of phase space."[11] In our one-dimensional setting it might seem that there is little variety to be had in attractors of dimension less than one--just limit cycles of p points and simple equilibria seem possible. Such is not the case. Non-linear first-order

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[9]Devaney, 24-7.

[10]Holden, A. V. and M. A. Muhamad, "A Graphical Zoo of Strange and Peculiar Attractors," 15-6: in Chaos, ed. A. V. Holden.

[11]Holden, A. V. and M. A. Muhamad, "A Graphical Zoo of Strange and Peculiar Attractors," 16: in Chaos, ed. by A. V. Holden.

difference equations such as the quadratic we explore in part III can give rise to strange attractors of <u>non-integral</u> dimension between 0 and 1. These strange attractors are qualitatively distinct from anything generated with linear "evolution equations"; indeed, as we will see in Part II, the only attractors in linear dynamical systems are single attractive fixed points.

3. Discrete versus Continuous Models

It should be noted that in adopting difference equations like $X_{n+1}=aX_n+b$ as our "equations of motion," we have chopped time up into discrete bits. There are times t_1, t_2, t_3, etc. in such a mathematical system, but no $t_{3/5}$ or t_π . Such discrete models are sometimes directly pertinent to real world phenomena, but in other cases they do violence to genuinely continuous, flowing phenomena. The population dynamics of a species with a brief annual breeding season lends itself well to discrete models, while the trajectory of a projectile would be caricatured by such treatment.

Often discrete models are applied to essentially continuous phenomena in the hope that rough approximations of the true behavior will be generated. Economists often employ difference equations in their models, for instance, even though they are aware that economic activity does <u>not</u> occur in discontinuous bursts.[1] These approximations may be successful in certain contexts, but the potential for introducing gross error exists. A certain continuous dynamic model when translated into a perfectly analogous <u>discrete</u> model may mean the difference between simple equilibrium and chaos.

Robert Devaney provides us with the following example. Suppose P_t is the population of an animal species at time t, and L is the maximum population that the food supply can support in full.

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[1]Baumol, 151-2 and 279-82.

Under a continuous model of population change such as

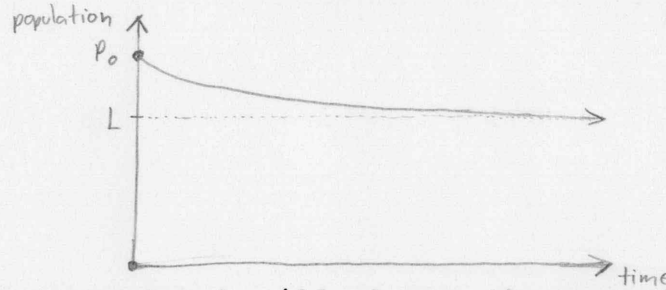
$$dP/dt = kP(1-P),$$

no matter what the "seed" population, P_t tends to L as $t \to \infty$. If

1. $P_t < L$ then $dP/dt > 0$;
2. $P_t = L$ then $dP/dt = 0$;
3. $P_t > L$ then $dP/dt < 0$.

Graphically, if P_0 exceeds L, then the pattern of adjustment to equilibrium is seen in Figure I.1.

Figure I.1



Varying the parameter k will change the curvature of the approach to equilibrium, but leaves the picture qualitatively unchanged.[2]

The analogous discrete dynamical system (called the "logistic map" for historical reasons):

$$X_{n+1}=k(X_n)(1-X_n)$$ [3,4]

gives rise to behavior which is radically different for certain k. In Devaney's words, "the dynamics of this system are still not yet completely understood."[5] The sequence P_n does converge to a simple equilibrium for certain k, albeit in the manner of Figure I.2.

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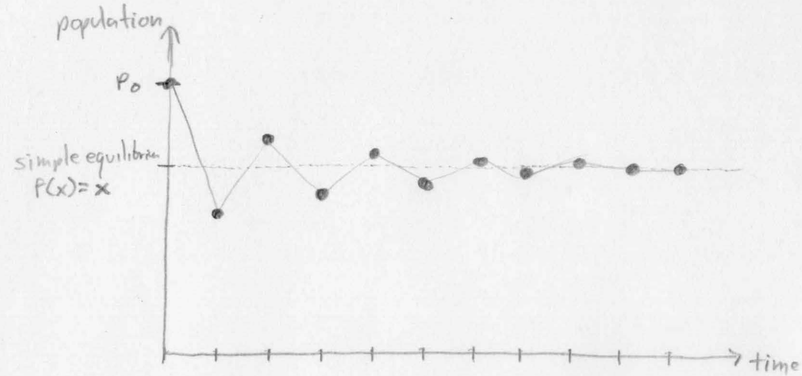
[2]Devaney, 3-6.

[3]Here L has been replaced by 1 for technical reasons--we can think of P_t now representing a fraction of maximal population.

[4]The logistic equation will serve as a prototypical quadratic difference equation in Part III.

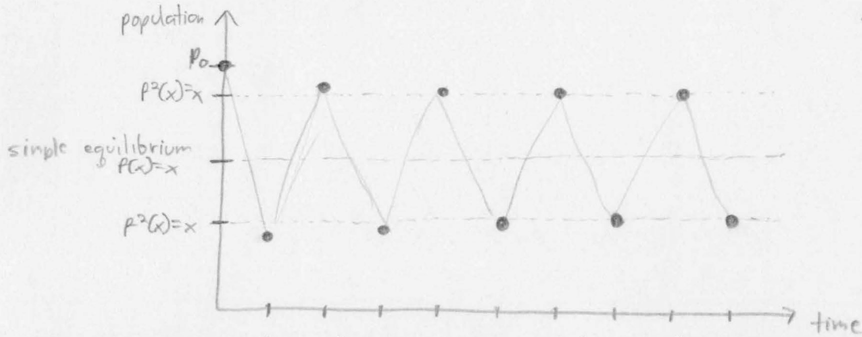
[5]Devaney, 6.

Figure I.2



For other values of k, however, P_n may tend to more complicated attractors such as a 2-cycle (k=3.1), a 3-cycle (k=3.84), or even to a strange attractor (k=4). A 2-cycle attractor, for instance, would cause the pattern seen in Figure I.3.[6]

Figure I.3



The bottom line is that one dimensional, continuous dynamical systems can have attractors no more complicated than simple equilibrium, while one dimensional, discrete systems can give rise to attractive p-cycles or even chaos.

Clearly there are hazards involved with the Procrustean use of discrete models for continuous phenomena, and vice versa. The prominence today of chaos theory has made many scientists wary of such errors, but some modern economists, for instance, still employ discrete models as a matter of convenience (economic data are collected at regular intervals), rather than by deliberate choice after close scrutiny of the nature of the phenomena being modelled.

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[6]Percival, 199-202.

4. Determinism and Prediction

When one writes down a function as an "equation of motion" for a dynamical system one has created a <u>deterministic</u> system. No matter where in phase space x_k is, there is exactly one x_{k+1} produced by applying some map f to x_k. Thus, given any point in phase space as an seed value, the entire future trajectory is determined. The seed value x_0 yields to precisely one x_1, which in turn generates a unique x_2 and so forth.[1]

How can a strictly deterministic system be reconciled with the <u>chaotic</u> behavior manifested in certain cases? The answer lies in <u>sensitive</u> <u>dependence</u> <u>on</u> <u>initial</u> <u>conditions</u> (**SD on IC**) which is a property of certain non-linear dynamical systems. While an exact x_0 gives an exact x_1 and so forth, a seed value "close" to x_0 does not necessarily give a value "close" to x_1. In fact small errors can be quickly amplified--a phenomenon often referred to as the Butterfly Effect.[2] This stands in complete contrast to the conventional view that small errors necessarily lead to small errors.[3] The physicist Maxwell put it thusly in 1873:

> ...from the same antecedents follow the same consequents....But it is not much use

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[1]Ekeland, 20-1.

[2]The Butterfly Effect states that a butterfly's decision to flap its wings in South America may lead to a storm a month later in Texas; see Gleick, 11-31.

[3]Ekeland, 64-7.

> in a world like this, in which the same antecedents never again concur, and nothing ever happens twice....The physical axiom that has a similar aspect is that "from like antecedents follow like consequents." But here we have passed from sameness to likeness....There are certain classes of phenomena...in which a small error in the data only introduces a small error in the result...The course of these events is stable. There are other classes of phenomena which are more complicated, and in which cases of instability may occur.[4]

These "other" classes involve non-linearities in the laws governing evolution over time.

The impact of the Butterfly Effect upon prediction is profound. Even if a real world phenomena could be captured perfectly by a dynamical model which involved **SD on IC**, long-range prediction would still be impossible. The most minute error in measurement of the state of the system at t=0 will cause cascading errors as time progresses. Long-range prediction is damned from the outset (in systems exhibiting **SD on IC**) by the fact that <u>all measurements are approximations</u>--Heisenberg ensures this.[5] As Ekeland notes,

> If the experimenter reproduces exactly the same initial conditions, he will observe exactly the same trajectory: this is what it means for a system to be deterministic. But in practice, the initial conditions can never be reproduced exactly. There has to be some discrepancy...unnoticed at the beginning...(which) will be amplified with time, resulting in the long run in a completely different situation. The system

[4]Quoted in Ekeland, 67.

[5]Gleick, 18-21.

> thus exhibits some degree of randomness: the (seemingly) same initial conditions may lead to significantly different evolutions.[6]

Due to this "randomness," prediction must take an attenuated form, contenting itself with probabilistic statements.

Some scientists in the field of chaos have seized upon the word random, and have begun to make declarations about free will and so forth.[7] This is truly a case of the tail wagging the dog. The **modus operandi** of such people is to investigate the behavior of an arbitrary mathematical construct, then to spell out the implications of the model for reality. Even so, such backwards reasoning should lead to the conclusion that "reality is deterministic because my model is deterministic." It is mere hubris which causes them to confuse "deterministic" with "determinable by man", and leads them to talk about free will. Chaos is not metaphysics. If there is a philosophic lesson to be learned, it is that if indeed the universe is deterministic, and involves phenomena which are **SD on IC**, then we as human observers are epistemologically barred from seeing the path of evolution in full. We may be able to predict tomorrow's weather, or even that of next week, but under no circumstances will we be able to predict the weather arbitrarily far into the future.

In the dynamical models we consider in Parts II and III, that is, one dimensional systems with both linear and non-linear maps,

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[6]Ekeland, 66-7.

[7]Ford as quoted in Gleick, 306. Also Crutchfield, 57.

the future is always deterministic. The future is contained in the present to paraphrase Ekeland.[8] The past, however is not always determined by the present state of affairs. In cases where the map f is 1-1, i.e., f^{-1} exists such that

$$f@f^{-1}=I=f^{-1}@f,$$

the past is unambiguously determined. Included here are all the linear maps of Part II--with f^{-1} we may look backward in time as easily as forward with f. When f is not 1-1 the past is not determined by the state of the present. The history of (x_n) is ambiguous and best called prehistory. All the quadratics in Part III fall in this category. If $x_0=9$ and $x_{n+1}=(x_n)^2$ then all we can deduce about x_{-1} is that it may be -3 or 3. Of course, in this case -3 is a dead end, but in general quadratic evolution equations like the logistic equation create an infinite tree of possible historical paths. Some entries in the tree may be equivalent, but nonetheless it is apparent that even in the $x_0=9$, $x_{n+1}=(x_n)^2$ model there are an infinite array of possible histories. In any case the past is hidden whenever a non-linearity in the evolution equations causes multiple points to be mapped to the same point.[9,10]

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[8]Ekeland, 16.

[9]For a few special points the past may be determinate in such cases: $X_0=0$ implies that $X_{negative\ n}=0$ just as $X_{positive\ n}=0$ in the "squaring" system above.

[10]Not all non-linear difference equations are not 1-1: $X_{n+1}=(X_n)^3$ for example.

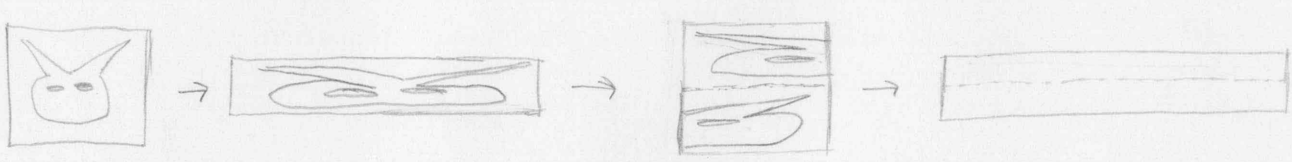
5. Arnold's Cat and the Taffy Pull Paradigm

> The key to understanding chaotic behavior lies in understanding a simple stretching and folding operation, which takes place in phase space.[1]

The effective randomness of dynamical systems which were chaotic required a new paradigm. It had to account for the "deterministic but random" behavior of systems which defied human prediction through sensitivity to initial conditions.

The new paradigm found its image in the actions of a baker who stretches and folds dough over and over again. For historical reasons, a cat (Arnold's Cat--Arnold being a Russian dynamicist) usually appears on the dough to give substance to the procedure.[2] After only a few stretches and folds, "Arnold's cat (is)...turned into mincemeat,"[3] as we see in Figure I.4.[4]

Figure I.4



Parts of Arnold's cat originally right next to each other are now

[1]Crutchfield, 51 as quoted by Dewdney, 109.

[2]Ekeland, 48-58.

[3]Ekeland, 50.

[4]Diagram I.4 is a reproduction of one in Ekeland, 51.

scattered by this deterministic process until total disorder seems to reign.[5,6]

In linear dynamical systems such as we consider in Part II there is no "stretching and folding" process--only stretching. The mathematical analog of "stretching and folding" <u>does</u> occur in certain nonlinear systems, however. For clarity we will examine a particular evolution equation, namely the logistic map f_k

$$X_{n+1}=k(X_n)(1-X_n)$$

with parameter k=4 to illustrate this.

Graphically the logistic map (with k=4) appears as in Figure I.5. For the present discussion we will worry only about the interval [0,1], noting that [0,1] is mapped onto [0,1] by f_4.

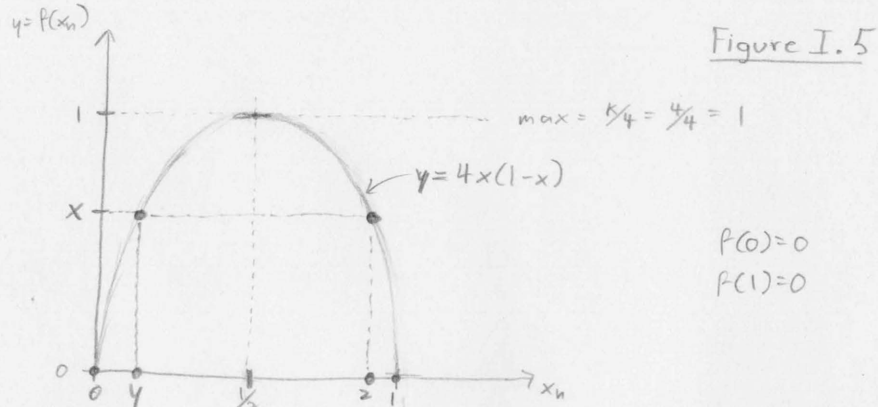


Figure I.5

It is clear that for any x in [0,1] except x=.5 there are exactly two values y and z which are mapped onto x. In fact [0,.5] is mapped onto [0,1] as is [.5,1]: the lower half of the unit interval is stretched to fit over [0,1], as is the upper half of the unit interval. <u>But this is to say that the unit interval is</u>

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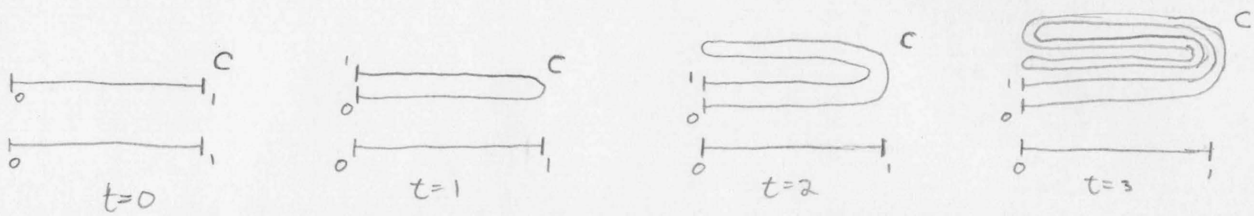
[5]Ekeland, 52.

[6]Conrad, M., "What is the Use of Chaos," 3: in <u>Chaos</u>, ed. A. V. Holden. Conrad suggests that a rotating taffy puller is perhaps even a better image for the "stretching and folding" operation.

stretched and folded. If we continue the process depicted in I.6 we see that, qualitatively,

Figure I.7



the unit interval is indeed stretched and folded in a manner analogous to the "baker's shift." In each case the curve C is a distorted unit interval which indicates how arbitrarily close points in [0,1] become torn apart from one another as time progresses.[7] An imperceptible error in measurement of x_0, the initial condition, asserts itself as seeming randomness as time passes.

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[7]Gleick, 50-2.

6. Discrete Dynamical Systems and Computers

Discrete dynamical systems lend themselves to exploration by computer. It is very helpful in generating conjectures to actually set up an iterative procedure to trace the orbits of points in phase space and merely observe. Pure abstraction which dominates nearly all of mathematics is here supplemented with methods that are literally experimental.[1] As Hofstadter notes in his essay Chaos and Strange Attractors, "not all mathematicians approve"[2] of this.

In practice, the behavior of a dynamical system with seed value x_0 is tracked by a simple procedure like the following:

```
INPUT X0
LOOP M TIMES over
  X=F(X)

LOOP P TIMES over
  X=F(X)
  DISPLAY X
```

where the M is sufficiently large to weed out the transients associated with the seed x_0, and bring the orbit infinitesimally near the attractor, if indeed there is one. By iterating P more times and displaying x_n on each pass we are then able to "see" the attractor (P must be chosen large enough to flesh out the

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[1]Hofstadter, 364-6.

[2]Hofstadter, 366.

attractor.)[3] If $x_n = x_k$ for all n>k then the attractor is a one-cycle; if $x_{n+q} = x_n$ for all n>k then the attractor is a q-cycle. If no pattern is apparent then one has chaos.

As a final note, it should be mentioned that a digital computer will introduce round-off error at each step of the iterative process outlined above. This poses no problem when the attractor is a simple equilibrium or a q-cycle. In those cases there is no sensitivity to initial conditions. When there is a chaotic (i.e., fractal) attractor, however, the accompanying **SD on IC** magnifies the computer round-off error into a significant factor. The result is that the fine, self-similar structure of the fractal attractor is completely obscured. One sees only randomness, and must translate this as chaos.

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[3]Dewdney, 108-9.

Part II: Linear First-Order Difference Equations

To set up a contrast for Part III we look at the simple dynamics on **R** produced by <u>linear</u> first-order difference equations, i.e., equations of the form

$$X_{n+1}=aX_n+b,$$

where a,b are both in **R**, and $a\neq 0$.[1] These dynamics are straightforward and will be exhaustively treated in a few pages. All of Part III, on the contrary, will cover only the rudiments of the dynamics of <u>quadratic</u> first-order difference equations.

To begin, we ask which points in **R** are fixed under the mapping $f(x)=ax+b$. We solve $f(x)=x$. Since $f(x)=ax+b$ we have $ax+b=x$ or

$$x=b/(1-a).$$

Thus we find that precisely one point in **R**, namely $x_{fixed}=b/(1-a)$, is <u>fixed</u> under the mapping f, unless a=1. If a=1 then our difference equation is of the form

$$X_{n+1}=X_n+b$$

and clearly all points in **R** are fixed under f if b=0, and no point whatsoever is fixed if $b\neq 0$.

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[1]If a=0 then we do not have a first order difference equation, merely a trivial dynamical system wherein $X_n=b$ for all $n\in N$.

Next we examine first-order linear mappings for cyclical points of order p where p>1. Are there any elements of **R** which are not fixed (i.e., $f(x)\neq x$), yet

$$\underbrace{f(f(\ldots f}_{n\ \text{times}}(x)\ldots))=x$$

Our first step is to look for x in **R** which are cyclical of order 2, but not fixed. We solve $f(f(x))=x$ or $f^2(x)=x$. Hence we have $a(ax+b)+b=x$ or

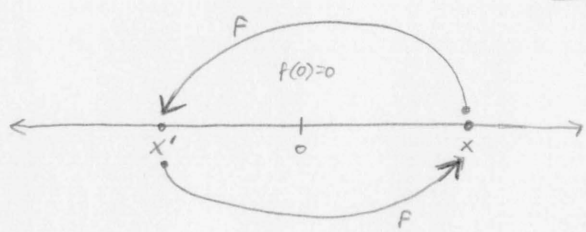
$$x=\frac{b}{1-a}\cdot\frac{1+a}{1+a}=\frac{b}{1-a}$$

Recall that $x_f=b/(1-a)$ is only a trivial 2-cycle as it is a fixed point.[2] Obviously, unless a=-1 the term (1+a)/(1+a) disappears leaving us with no non-trivial solution. If a=-1 then our evolution equation is of the form

$$X_{n+1}=-X_n+b$$

and every point in **R** is 2-cyclic, satisfying $f(f(x))=x$: in addition one of these points is fixed. Figure II.1 illustrates the workings of the map $X_{n+1}=-X_n+b$ where b=0.[3]

Figure II.1



Each point x in **R** is mapped to its mirror image x' (the "mirror" is b=0), which in turn is mapped back to x.

Thus in one very special case (a=-1) our linear evolution

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[2]A fixed point, also called a 1-cycle, is trivially a p-cyclic point of any order p.

[3]Other b give qualitatively identical pictures.

equation creates the 2-cycles of Figure 1. Every x in **R** is trapped on a 2-cycle from the very outset--there is no attracting and no repelling involved.

This is the high water mark of one dimensional first-order linear dynamics. There are no other limit cycles to be found. We prove below that such dynamical systems possess no (non-trivial) p-cycles for $p>2$.

To show this we first define homeomorphism. A homeomorphism of **R** is a function $f: \mathbf{R} \to \mathbf{R}$ such that f is 1-1, onto, continuous, and f^{-1} is also continuous.[4] Clearly our linear evolution equations satisfy these conditions. Now we prove that a homeomorphism of **R** has no (non-trivial) cyclic points of period greater than two.

Theorem 1 A homeomorphism $f:\mathbf{R}\to\mathbf{R}$ has no (non-trivial) points of period three or greater.

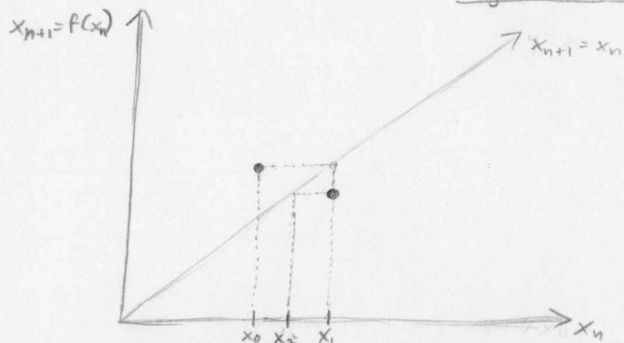
Proof Since f is 1-1 and continuous, f must be either strictly decreasing, or strictly increasing.

Suppose f is strictly increasing ($x<y$ implies $f(x)<f(y)$), and further suppose that $f(x_0)=x_1>x_0$. It follows that $x_0<x_1<x_2<...<x_n$. Clearly $f^n(x_0)$ will never return to x_o under these conditions. If we assume that $f(x_0)=x_1<x_0$ then again $f^n(x_0)$ never returns to x_0 as $x_0>x_1>x_2>...>x_n$. The only remaining case is that where $f(x_0)=x_0$ in which case we have a simple 1-cycle.

[4]Devaney, 9.

Now take f to be strictly decreasing ($x<y$ implies $f(x)>f(y)$). As above, if $f(x_0)=x_0$ then x_0 is of period 1 and we are through. If we assume $x_1>x_0$ then $f(x_1)=x_2<x_1=f(x_0)$. Suppose $x_2>x_0$ as shown in Figure II.2. Under these conditions all x_k for $k>2$ lie between x_0 and x_1 and thus $f^n(x_0)$ never returns to x_0. This statement is justified because if $x_0<y<x_1$ it follows that $f(x_0)=x_1>f(y)>x_2=f(x_1)>x_0$.

Figure II.2



If $x_2=x_0$ then x_0 is of period 2. Finally if $x_2<x_0$, then, since $x<y$ implies $f(f(x))<f(f(y))$, we see that the subsequence $(x_1, x_3, x_5,...)$ is increasing and the subsequence $(x_2, x_4, x_6,...)$ is decreasing. But this means that $f^n(x_0)$ never returns to x_0 as we know that $x_2<x_0<x_1$.

The other case where $x_1<x_0$ proceeds similarly.

Now we turn to the question of whether the fixed points of $X_{n+1}=aX_n+b$ are <u>attractive</u> or <u>repellent</u> equilibria or neither. Immediately, we dispense of the pathological cases where $|a|=1$. If $a=-1$ then nothing can be added to the analysis above. If $a=1$ and $b=0$ then every point in **R** is a fixed point and its own trivial attractor. The time path of any seed value x_0 is simply $(x_0, x_0, x_0,...)$. If $a=1$ and $b\neq 0$ then the dynamics are as shown in Figure II.3.

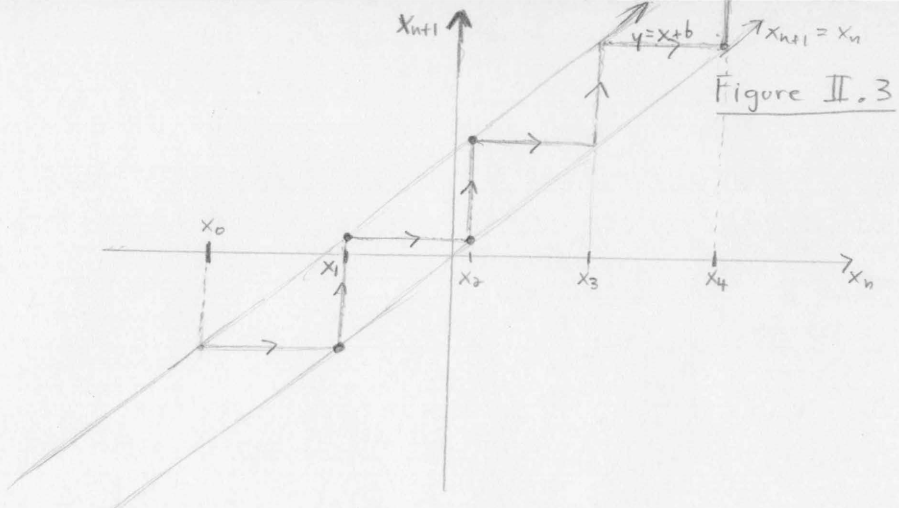


Figure II.3

This picture introduces the "45-degree diagram" that will play an important role in Part III.[5,6] It is evident from the figure that $(x_n)\to\pm\infty$ depending upon the sign of b. An algebraic proof of this merely requires that we recognize (x_n) as an arithmetic sequence.

This leaves us with only the well behaved cases where $a\neq 1$. Here there is always a unique fixed point $x_f=b/(1-a)$.

There are two distinct dynamical behaviors to be found. If $|a|>1$ then x_f is <u>repellant</u>, while if $|a|<1$ then x_f is an <u>attractive</u> equilibria. The basin of influence is all of **R** in both cases. Figures II.4 a,b,c,d show the dynamical behavior of (x_n) for arbitrary seed values given different values of a, respectively a>1, 1>a>0, 0>a>-1, and a<-1.

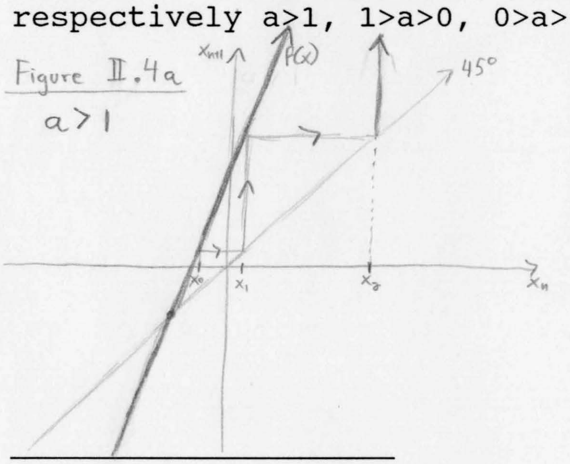


Figure II.4a
a>1

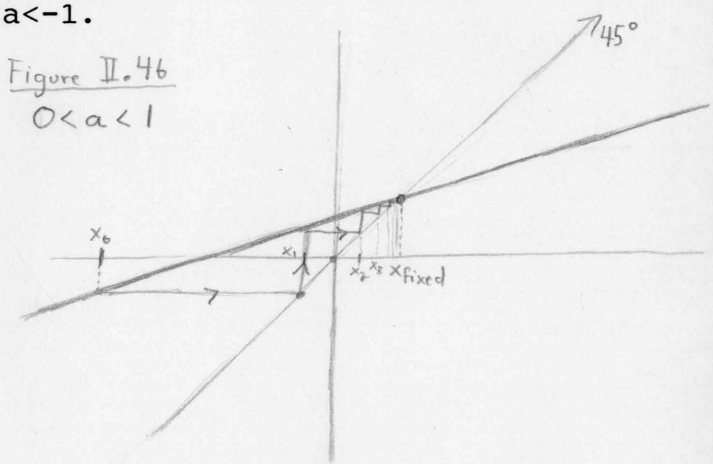
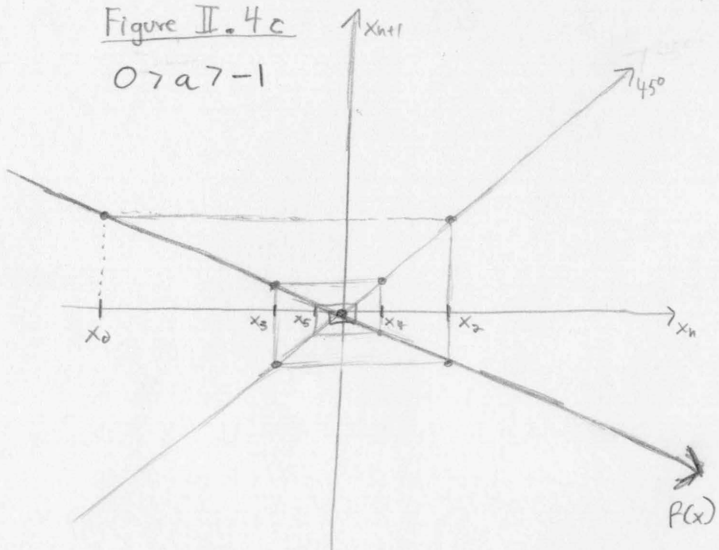
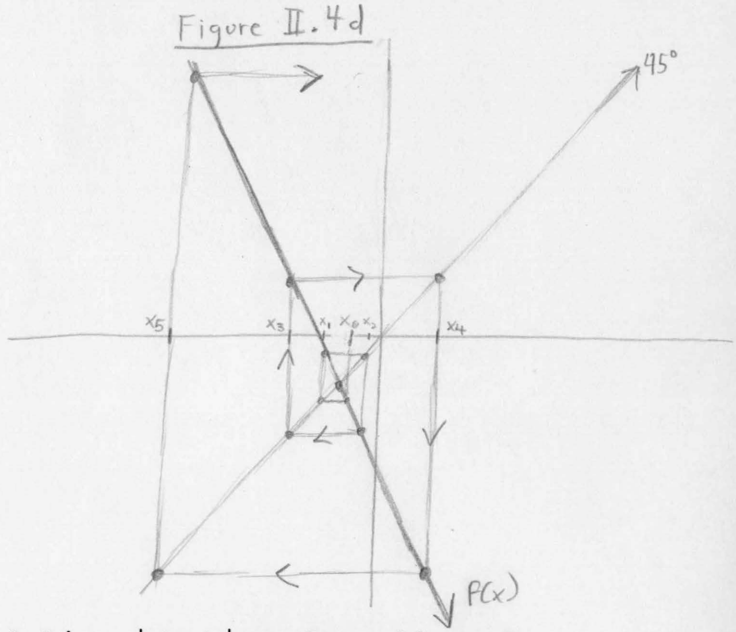


Figure II.4b
0<a<1

[5]This diagram plots x_{n+1} versus x_n to allow a visualization of the time path of (x_n); the 45° line is used to translate points from one axis to the other.

[6]Diagram employed in Gleick, 176.





If $|a|>1$ then x_f is repellant and $\lim_{n\to\infty}|(x_n)|=\infty$ for all $x_0\epsilon\mathbf{R}$ other than x_f. If $|a|<1$ then x_f is an attractive fixed point, and $\lim_{n\to\infty}(x_n)=x_f$ for all $x_0\epsilon\mathbf{R}$.

The time path of (x_n) can be described succinctly in algebraic terms for those unconvinced by the diagrams above. We can literally solve for x_n in terms of x_0 and n due to the simple linear nature of the evolution equation:

$$x_1=ax_0+b$$
$$x_2=a(ax_0+b)+b=a^2x_0+b(1+a)$$
$$x_3=.........=a^3x_0+b(1+a+a^2)$$

and by induction,

$$x_n= a^nx_0 +$$

if we ignore the pathological case where a=1.[7]

Given this equation it becomes apparent that if $|a|<1$ then as $n\to\infty$ the highlighted terms become infinitesimal and (x_n) approaches $x_f=b/(1-a)$. Similarly if $|a|>1$ then $\lim_{n\to\infty}|(x_n)|=\infty$. With this we have exhausted the dynamical possibilities of the family of linear one dimensional first-order difference

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[7]Goldberg, 63.

equations. Their simplicity makes for a striking contrast with the rich dynamical behavior of the non-linear families[8] that we deal with in Part III.[9,10]

[8]The logistic family and the family $X_{n+1}=(X_n)^2-a$.

[9]Goldberg, 63-87 investigates the material of Part II from a different angle which the reader may find helpful.

[10]The reader is referred to Appendix B on Catastrophe Theory to see how that applies to the above analysis.

Part III: Quadratic First-Order Difference Equations

1. Introduction

In this section we examine non-linear dynamics through the logistic family of maps $X_{n+1}=f_k(X_n)=k(X_n)(1-X_n)$, or where more convenient, through the family of mappings $X_{n+1}=f_a(X_n)=(X_n)^2-a$. [1] All of the dynamical properties exhibited by one are also present in the other--moreover there is a whole class **S** of non-linear evolution equations which possess dynamical properties qualitatively identical to that of our prototypical quadratics. If there exists a "k" such that the logistic map yields an attractive 3-cycle then there exists a corresponding parameter value "a" such that $X_{n+1}=(X_n)^2-a$ exhibits an attractive 3-cycle, for instance.

The conditions that must be met by an evolution equation for it to fall into this class **S** are minimal. There must be an interval I in **R** which is mapped into itself by the continuous function f such that f has a unique maximum--to the left of which f is strictly increasing, and to the right of which f is strictly decreasing.[2] These conditions are met by the logistic function in the interval [0,1] when 0<k<4 as shown in Figure III.1.

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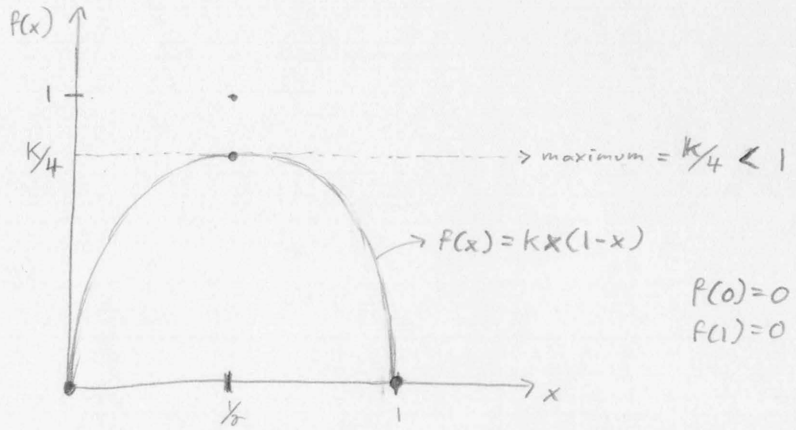
[1]The two families of functions are $f_k=kx(1-x)$ and $f_a=x^2-a$.

[2]Preston, 1-2.

Figure III.1



Families of iterative functions as diverse as $X_{n+1}=rX_n^{(1-rx)}$ and $X_{n+1}=r\sin(\pi X_n)$ satisfy these conditions for certain ranges of the parameter r, and exhibit behavior topologically equivalent to that generated by the logistic family for $0<k<4$.[3]

These facts are established in Devaney's Introduction to Chaotic Dynamical Systems in parts I.6 and I.7 on symbolic dynamics and topological conjugacy. They are outside the scope of this paper, but we shall use them as needed.

In Figure III.2 is a bifurcation diagram for the family of evolution equations $X_{n+1}=(X_n)^2-a$.[4,5] On the horizontal axis is the parameter "a" which is crucial to the behavior of the iterates of (x_n), and on the vertical axis are plotted the points of accumulation of (x_n).[6] There is vast array of behavior visible

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[3]As might be expected, a family of maps like $X_{n+1}=(X_n)^2-a$ satisfies the obverse of our conditions (a unique minimum rather than a unique maximum, etc.) and also exhibits dynamical behavior that is topologically identical to that of the logistic family.

[4]Diagram produced by Dana Borger, NSF.

[5]See Appendix C for a further description of bifurcation diagrams.

[6]The range of "a" displayed in the bifurcation diagram is [1/4,2] because this is the range of interesting dynamics for this family.

a = -0.25000000 to 2.00000000 XI = 0.5000 Xmag = 2.50

Figure III.2

Bifurcation Diagram for the family of mappings $x_{n+1} = x_n^2 - a$

region $-\frac{1}{4} < a < 2$

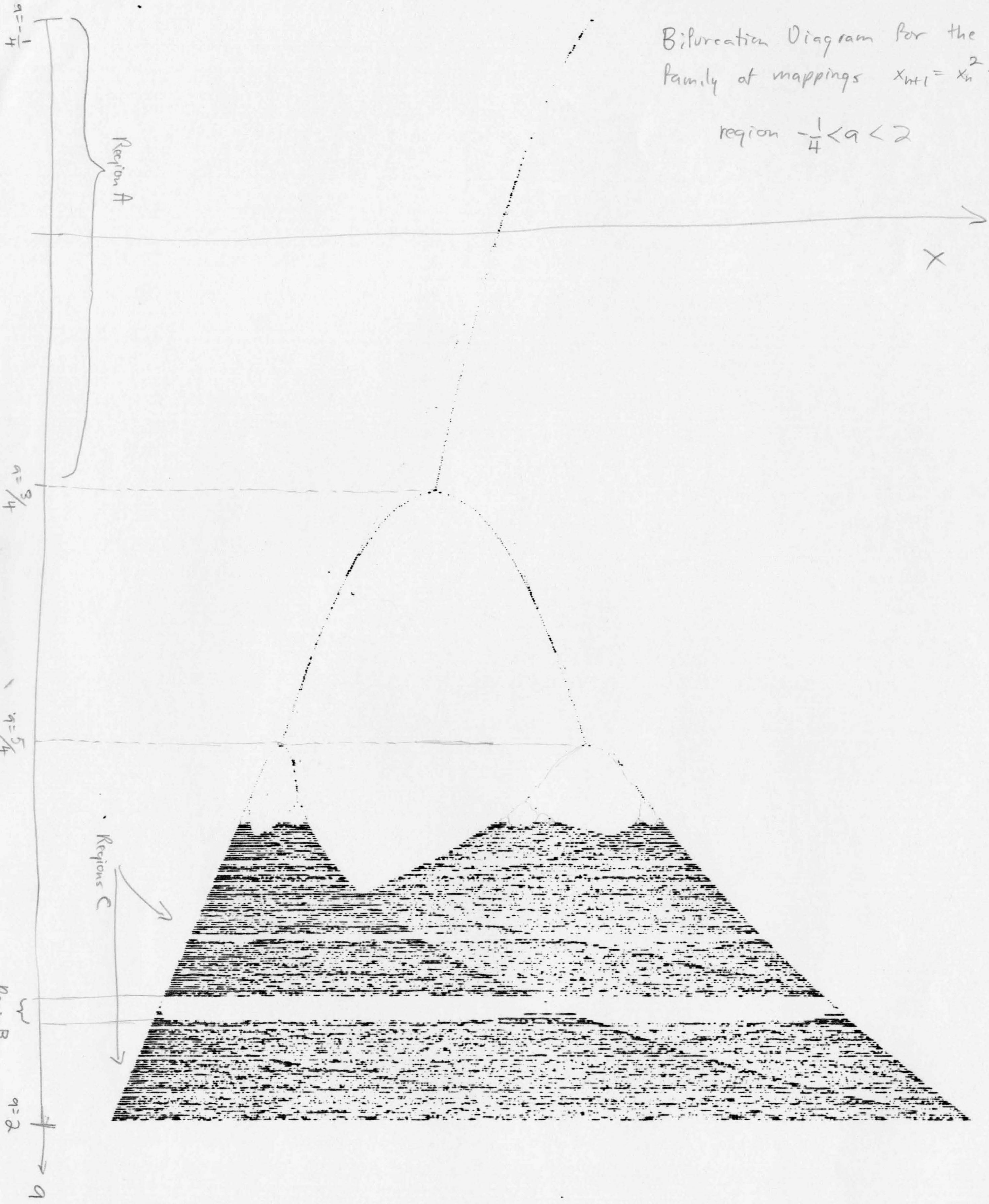


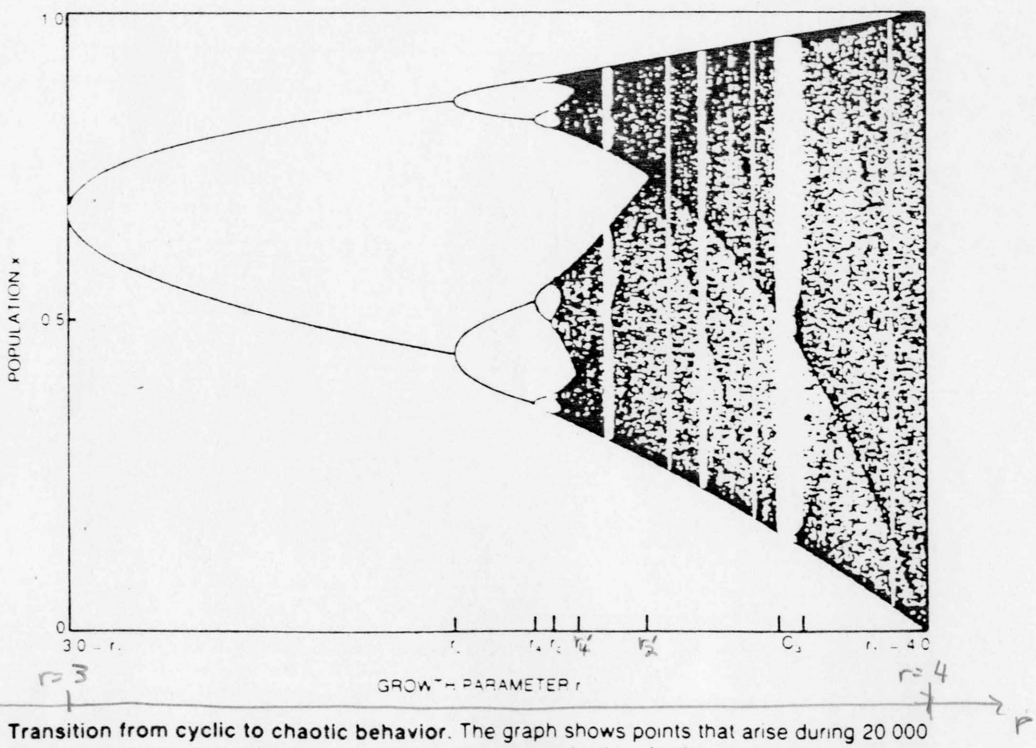
Figure 4

including a region of "a" which causes x_n to converge to a single value (Region A), a region with an attractive 3-cycle (Region B), and even areas of seeming randomness such as Region C. We will only begin to explore the complexities of this diagram in the pages to follow.

With the presentation of the bifurcation diagram the statement about the topological equivalence of the families of iterative equations can be strengthened. The bifurcation diagram of the logistic family can be stretched and contracted to look just like that for $X_{n+1}=(X_n)^2-a$ or any other family in **S**. This surprising fact is supported by a second bifurcation diagram (Figure III.3), this one for the logistic family.

Figure III.3

Bifurcation Diagram for Logistic Mappings $x_{n+1} = r(x)(1-x)$
for the region $3 < r < 4$



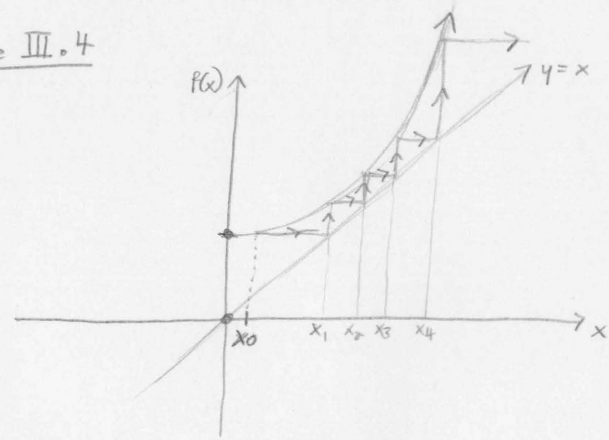
Transition from cyclic to chaotic behavior. The graph shows points that arise during 20 000 iterations in computing values of population for some initial value. As the growth parameter r increases from 3 to r_c the population oscillates among 2, 4, 8, . . . 2^n . . . values. At r_c the infinity of lines becomes an infinity of bands, the values of the population oscillate in a regular fashion among the bands, but take on random values within each band. As r increases above r_c, the bands merge, until for values of r above r_2 there is only a single band of values that the population assumes chaotically. The thin white stripes, such as the region labeled C_3 represent periods in which the population assumes regular values for much of the time and is only intermittently chaotic. While the scale in figure 4 is highly nonlinear, the scale here is linear. Figure 5

2. The Region of Interest

The task of examining a family of non-linear dynamical equations on all of **R** is not as ominous as it may sound. Only a small region of parameter values "b"[1] give interesting behavior. The remaining areas can be handled in short order. Similarly, for all but a small region of seed values, (x_n) simply spins madly off to $\pm\infty$.

The family of maps $X_{n+1}=(X_n)^2-a$ for instance has "interesting" dynamical behavior only when $-1/4<a<2$. If $a<-1/4$ then regardless of x_0, $\lim_{n\to\infty}(x_n)=\infty$.[2] Figure III.4 below indicates the behavior of arbitrary x_0 under iteration when $a<-1/4$.

Figure III.4



This can be proven with little effort.

Theorem 2 Let $X_{n+1}=(X_n)^2-a$. If $a<-1/4$, then $\lim_{n\to\infty}(x_n)=\infty$ for all $x_0\in\mathbf{R}$.

---

[1]We employ "b" as a generic function parameter.

[2]There are no fixed points as $f(x)=x$ has no real solutions for $a<-1/4$.

Proof We show that (x_n) is increasing, and that (x_n) is unbounded above. To show increasing we note that

$$x_{n+1}-x_n = ((x_n)^2-a)-x_n = ((x_n)^2-x_n)-a \geq -1/4-a > 0$$

since $a<-1/4$. But then $x_{n+1}>x_n$ for all $n\epsilon\mathbf{N}$.[3] To show unbounded we observe that if (x_n) were to converge to a point p, it would necessarily be the case that (x_{n+1}) converges to $f_a(p)>p$ since $f(x_n)=x_{n+1}$. This gives an immediate contradiction since (x_n) and (x_{n+1}) must have the same limit.[4] With (x_n) both increasing and unbounded we are done.

In a similar manner, if $a>2$ then "almost all" x_0 cause (x_n) to diverge to infinity. This cryptic statement must remain unproven until the end of this section.

We turn for a moment to a consideration of the ranges of x_0 that produce interesting dynamics. For each map f_a with $-1/4<a<2$ the range of interest is

$$-R= -\frac{(1+\sqrt{[1+4a]})}{2} < x_0 < \frac{(1+\sqrt{[1+4a]})}{2} =R$$ [5]

We call this interval J. If x_0 is in J, then the sequence (x_n) is forever trapped in J; if not, (x_n) diverges to infinity. We prove

Theorem 3 Let $x_{n+1}=(x_n)^2-a$. For $a>-1/4$, if $|x_0|>R$ then $\lim_{n\to\infty}(x_n)=\infty$.

---

[3]Note that the minimum of $f(x)=x^2-x$ is $-1/4$.

[4]This brief proof of unboundedness is due to Devaney, 32.

[5]"R" is the larger root of $f(x)=x^2-a=x$; "r" is the smaller root.

<u>Proof</u> Again we show (x_n) to be increasing and unbounded. We proceed by induction to show increasing. Let $|x_0|=R+c$ where c is positive. Clearly x_1 is <u>positive</u> and exceeds x_0 for $a>-1/4$ since

$$x_1=f(x_0)= \frac{1+\sqrt{[1+4a]}}{2}+c+c^2+c\sqrt{[1+4a]} = |x_0|+c^2+c\sqrt{[1+4a]}.$$

By similar means it can be shown that $x_2>x_1>0$.

With this established we show that $x_k>x_{k-1}>0$ implies $x_{k+1}>x_k$.

$$x_k > x_{k-1}$$

$$(x_k)^2 > (x_{k-1})^2$$

$$(x_k)^2-a > (x_{k-1})^2-a$$

$$x_{k+1} > x_k$$

Thus (x_n) is increasing. To show unboundedness we repeat the argument given in the proof of Theorem 2.

With this we have a well defined block of "a" and "x_0" values that produce interesting dynamical behavior--Figure III.5 shows this region for the family $x_{n+1}=(x_n)^2-a$. Note that since J depends upon "a" the region is not square.

$$|R|=\frac{(1+\sqrt{[1+4a]})}{2}$$

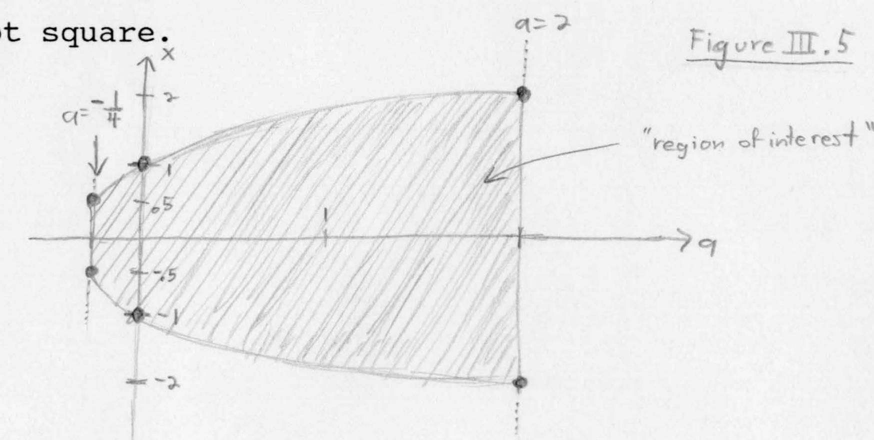
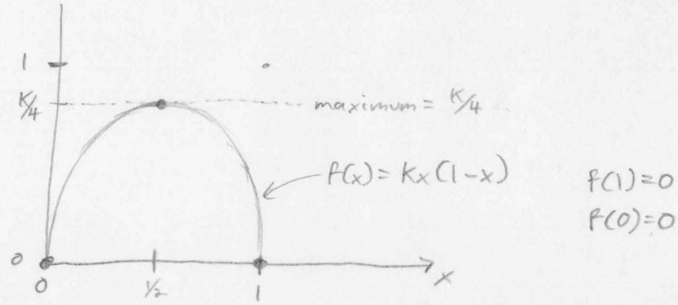


Figure III.5

For the logistic family there is a corresponding picture stretching from k=0 to k=4 with one major difference. The interval J of seed values that are trapped under iteration is always the unit interval [0,1]. Thus the "region of interest" is a simple rectangle.

To complete our treatment of the "region of interest" we examine the logistic mapping shown in Figure III.6 for k>0.

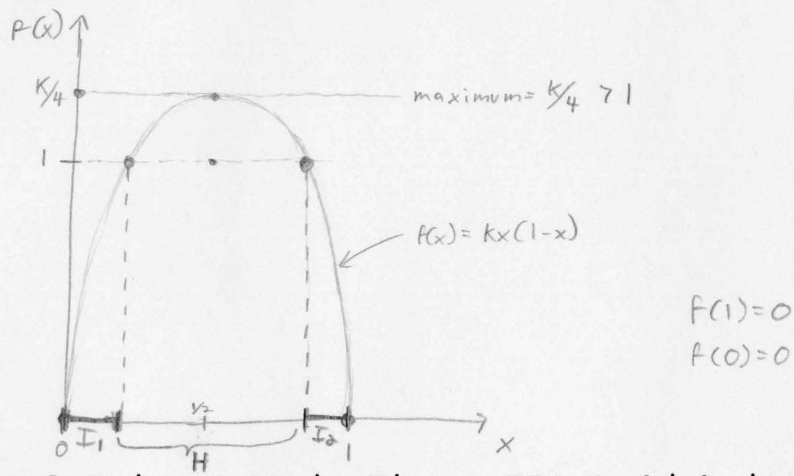
Figure III.6



with a maximum at x=1/2 of height k/4. Clearly, so long as 0<k<4 the unit interval is mapped into itself.[6]

The value k=4 for the logistic family plays the same role that a=2 does for the $x_{n+1}=(x_n)^2-a$ family. Due to the simplicity of the logistic family in this context, we will address the earlier statement that "almost all x_0 cause divergence to infinity if a>2" by looking at k>4 for the logistic.

If k>4 then Figure III.7 applies.



There is an interval H in [0,1] in Figure III.7 which is mapped under f_k outside of [0,1] never to return. The two disjoint intervals I_1 and I_2 are <u>each</u> mapped ("stretched") onto all of the

[6]Devaney, 33.

unit interval [0,1]. With another iteration the process of Figure III.7 is repeated for each of $f(I_1)=[0,1]$ and $f(I_2)=[0,1]$. The centers of I_1 and I_2 are both mapped outside of [0,1] where they proceed to head off to infinity. The continuation of this process indefinitely leaves only a Cantor Dust[7] of points inside the unit interval which forever remain there. "Almost all" points will eventually jump outside the unit interval and spin off to $-\infty$.

The balance of Part III will examine the "region of interest" for the logistic family, or where more convenient, its cousin, the family $x_{n+1}=(x_n)^2-a$.

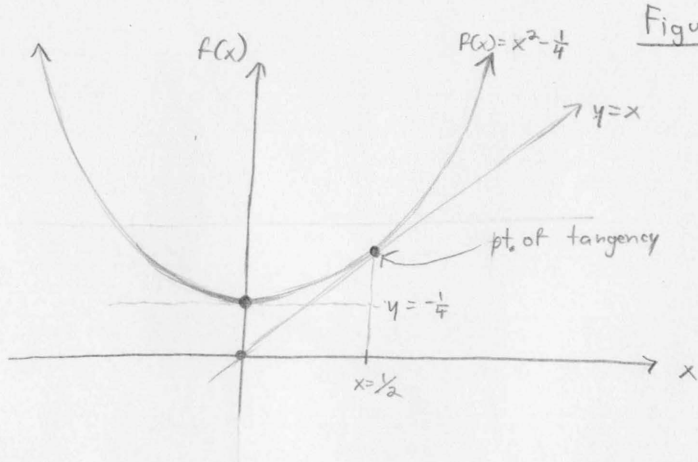
[7]See Appendix A.

3. Bifurcations and Universality

At the lowest parameter values shown in the bifurcation diagrams (Figures III.2 and III.3) there is simple convergence exhibited by (x_n). Then as the parameters ("k" or "a") increase, there is a trend toward increasing complexity until the chaotic regime is reached where no pattern whatsoever is discernible.

With this in mind we begin at the extreme left of Figure III.2 where a is at its lowest value, namely, -1/4. What happens when "a" passes the value -1/4 that causes (x_n) to converge under iteration where before it had not? The answer is to be found in the <u>tangent</u> <u>bifurcation</u> depicted in Figure III.8.

Figure III.8



The parabola representing $f(x)=x^2-a$ finally descends enough to intersect y=x when $a\geq-1/4$, giving <u>real</u> solutions to $f_a(x)=x$. At a=-1/4, the two curves are tangent and $f_a(x)=x$ has the double root x=1/2. Arbitrary x_0 in (-1/2, 1/2) converge to the fixed point x=1/2 as shown in Figure III.9.

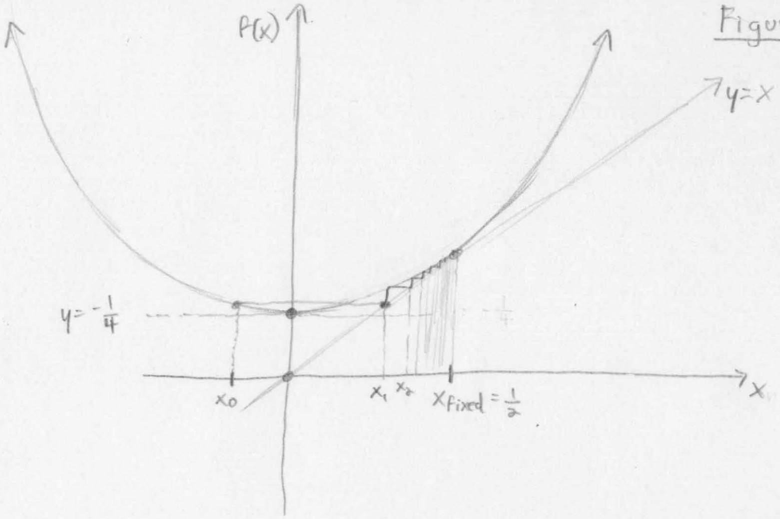


Figure III.9

For all a in the range [-1/4, 3/4] the equation $f_a(x)=x$ has two <u>real</u> solutions, one attractive and one repellant.[1] The larger solution "R" is repellant because $|f_a'(R)|>1$, while the smaller root "r" is attractive since $|f_a'(r)|<1$.[2,3]

Diagrams III.10a and III.10b illustrate the vast difference between iterative behavior near a fixed point where $|f_a'(x)|>1$, and behavior near a fixed point where $|f_a'(x)|<1$.

Figure III.10a

$|f_a'(x)|<1$

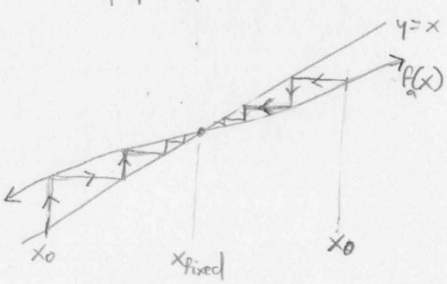
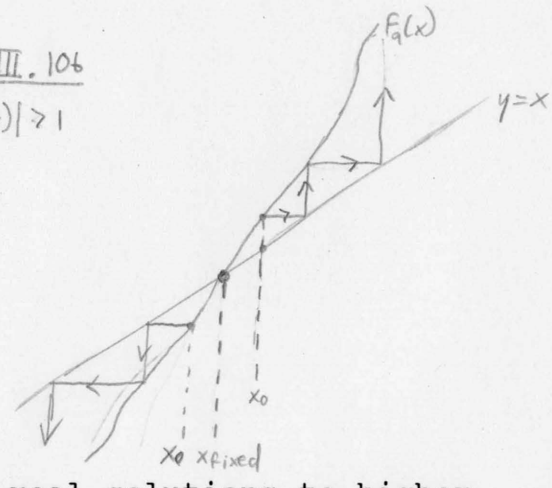


Figure III.10b

$|f_a'(x)|>1$



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[1]For this range of "a" there are no real solutions to higher degree equations $f_a^n(x)=x$ aside from the two trivial solutions which also satisfy $f_a(x)=x$.

[2]On the importance of the slope of f at x_{fixed} in determining attracting and repelling behavior the reader is referred to section II, especially Figure II.4.

[3]Where, as at a=-1/4, $|f'(x_{fixed})|=1$, the determination of attractive/repellant status is tedious and will not concern us. It so happens that all tangency bifurcations for $X_{n+1}=(X_n)^2-a$ will be attractive, as we see for a=-1/4.

Picard's Theorem formalizes what is shown in Figure III.10a. If $f: [a,b]\rightarrow[a,b]$ is continuous and differentiable on [a,b] such that $|f'(x)|\leq L<1$ for all x in [a,b] and x_0 is in [a,b], then the sequence (x_n) defined by $x_{n+1}=f(x_n)$ converges to the solution of $f(x)=x$ in [a,b]. This is proven in standard real analysis texts. In a similar vein, if $|f'(x)|\geq L>1$ at x_{fixed} then there is some interval [a,b] around x_{fixed} such that (x_n) will eventually be repelled out of [a,b] for all x_0 in [a,b], except $x_0=x_{fixed}$ of course.

As "a" passes 3/4, $|f_a'(r)|$ becomes larger than one, and "r" joins "R" as a repellant equilibrium. At the precise moment that "r" becomes unstable, however, an attractive 2-cycle appears. When $a\geq3/4$, the fourth degree equation $f_a(f_a(x))=x$, which previously had only the two trivial roots "r" and "R," develops two additional real roots. These two new roots (x_a and x_b) are, algebraicly,[4]

$$\frac{-1\mp\sqrt{1+4(a-1)}}{2} = \frac{-1\mp\sqrt{4a-3}}{2}$$

Since they are stable so long as $|f^{2'}(x)|=|4x^3-4ax|$ is less than 1, they remain attractive from a=3/4 until a=5/4. The value of $|f^{2'}(x)|$ at both x_a and x_b begins at 1 at a=3/4, then decreases past 0 till finally reaching -1 at a=5/4.

Before moving on, it is easily established that the value of $|f^{n'}(x)|$ is the same at all n points of an n-cycle. We show that if $(c_1, c_2, c_3,\ldots c_n)$ is a n-cycle under f, then

$$f^{n'}(c_1)=f^{n'}(c_2)=\ldots=f^{n'}(c_n)$$

[4]We factor $(x^2-a)^2-a$ to obtain $(x^2-x-a)(x^2+x-(a-1))$. The expression $(x^2+x-(a-1))$ yields our two new roots.

By use of the chain rule for derivatives

$$f^{n'}(x)=f'(f^{n-1}(x))*f'(f^{n-2}(x))*...*f'(f(x))*f'(x)$$

But then $f^{n'}(c_i)=f'(c_1)*f'(c_2)*...*f'(c_n)$ by mere rearrangement for all c_i in the n-cycle and we are done. An immediate corollary of this is that each branch of a 2^k-cycle splits at precisely the same moment.

The process seen above where an attractive 1-cycle split into an attractive 2-cycle is called a pitchfork bifurcation[5]. It is repeated endlessly. At a=5/4 an attractive 4-cycle appears, to be followed by an 8-cycle, then a 16 and so on.[6] These bifurcations come at ever shorter intervals and accumulate at a≈1.41 for our family of maps $X_{n+1}=(X_n)^2-a$. It should be emphasized that this bifurcation process is characteristic of all the families in **S**, i.e., of all "stretching and folding" type maps.

The bifurcation phenomenon is not only a qualitative regularity, however, but also universal in a quantitative sense. The physicist Mitchell Feigenbaum was the first to notice this in the mid-70's.

Feigenbaum found that the (attractive) 2^n-cycle regions became narrower in a very predictable manner for various families in **S**. In fact, he found that the range of the parameter b yielding an attractive 2^k-cycle approaches (roughly) 4.669 times that of the

---

[5]In contradistinction to a tangency bifurcation where an attractive cycle appears preceded by nothing.

[6]Closed form computations of the "a" values at which bifurcations occur quickly become impossible as the equation $f^n(x)=x$ is of degree 2^n.

succeeding 2^{k+1}-cycle as k increases.[7,8]

As a corollary of this universal convergence, there is an accumulation point for the 2^n-cycles called 2^∞ for all families in **S** that defines the border between orderly 2^n-cycles and the <u>chaotic regime</u>.

For the family $X_{n+1}=(X_n)^2-a$ we have the following data to support the above statements:

n	range	width	ratio of band widths
1	-.25 to .75	1.00	
			2
2	.75 to 1.25	.50	
			4.24
4	1.25 to 1.368	.118	
			4.53
8	1.368 to 1.394	.026	
			4.727
16	1.394 to 1.3995	.0055	

Beyond this point the data I have contains too much error to be of use, but clearly these results point to Feigenbaum's Number.[9]

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[7]Feigenbaum, 50-1: the constant 4.669... is called Feigenbaum's Number.

[8]Feigenbaum did not rigorously prove this--that came in 1979 by Oscar Lanford; Gleick, 183.

[9]This data was generated by Terry Estes and Dana Borger, NSF.

4. Windows

To this point we have looked at the procession of period doubling that occurs after the advent of a 1-cycle at the extreme left of the bifurcation diagram (at a=-1/4 for $X_{n+1}=(X_n)^2-a$). There are other non-chaotic regions, however, that abruptly appear in the chaotic regime and also exhibit period doubling. These "n-windows" of order amongst chaos appear at irregular intervals--casual inspection of Figure III.2 reveals several such regions.

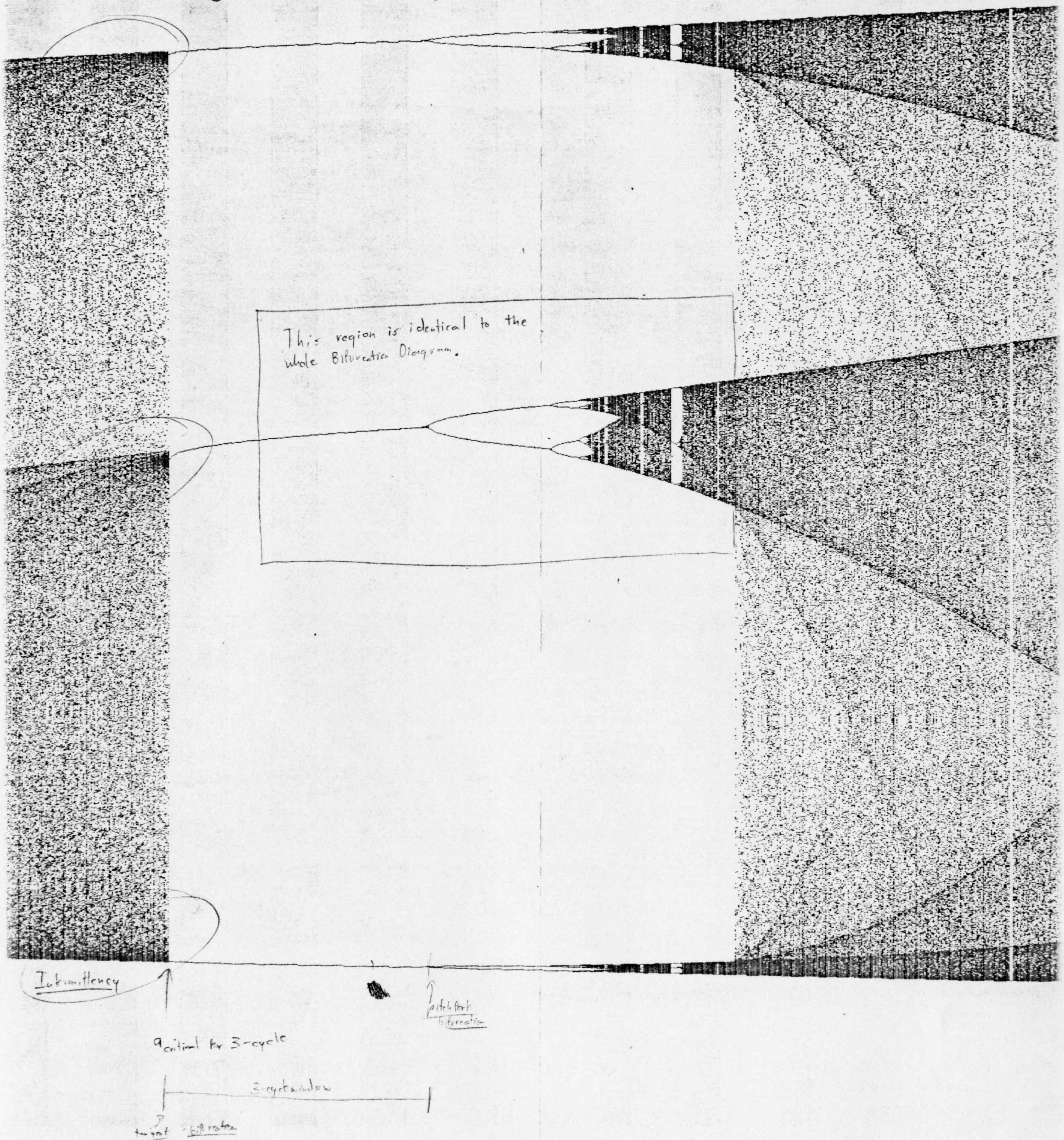
Each such n-window comes into being when the parameter "b" reaches a critical value such that $y=f_b^n(x)$ finally intersects y=x (simultaneously at n different points). Just as the 1-cycle began with a <u>tangent bifurcation</u>, so does each n-window embedded in the chaotic regime. The only difference is that each window begins with an n-cycle ($n\neq1$), instead of a 1-cycle.

As b increases beyond $b_{critical}$, the process of period doubling occurs as outlined in section III.3. Since the window began at its extreme left as an n-cycle, though, the sequence of periodicities is n, 2n, 4n, 8n,...rather than 1, 2, 4, 8....

Figure III.11 is a closeup of the 3-window for $X_{n+1}=(X_n)^2-a$. A close examination of one of the three arms of this picture reveals that the arm <u>replicates the entire bifurcation diagram</u>. In other words the bifurcation diagram is self-similar or fractal. One interesting result regarding these windows is that there exists a window of all orders p where p is prime. I can only prove this

Figure 6.11

Closeup of 3-window in $x_{n+1} = x_n^2 - a$ Bifurcation Diagram.



in the context of the $X_{n+1}=(X_n)^2-a$, but relying on the notion of topological conjugacy[1] it is clear that such a result applies to all classes of mappings in **S**.

Theorem 4 For every prime p there is an attractive p-cycle in the region $1/2<a<2$ under the iterative function $X_{n+1}=(X_n)^2-a$.

Proof We show that there exists an attractive n-cycle (n>1) with 0 as a member somewhere in the range $1/2<a<2$. Clearly if 0 is an element of an n-cycle, the n-cycle is attractive as:

$$f^{n'}(x)=f'(f^{n-1}(x))*f'(f^{n-2}(x))*...*f'(f(x))*f'(x)$$

$$f^{n'}(0)=..................................*2x=2(0)=0$$

Now consider the equation $f_a^n(0)=0$ for all n>1. Let $p_n(a)$ be the polynomial in "a"[2] that corresponds to $f_a^n(0)$. We show that $p_n(a)=0$ has at least one root in the region $1/2<a<2$, and thus that there is an "a" producing a stable n-cycle in $1/2<a<2$. Our procedure will be to prove $p_n(1/2)<0$ and $p_n(2)>0$ by induction for all n>1, which together imply that $p_n(a)=0$ for some real "a" in $1/2<a<2$.

1. $p_n(1/2)<0$:

For n=2 we have $f^2(0)=p_2(a)=a^2-a$

and so $p_2(1/2)=1/4-1/2=-1/4$.

Now assume $-1/2< p_k(1/2) <0$. Thus we have

$$0< p_k(1/2)^2 <1/4.$$

Subtracting 1/2 gives

$$-1/2< p_k(1/2)^2-1/2 <-1/4<0, \text{ or}$$

$$-1/2< p_{k+1} <0 \quad \text{as needed.}$$

---

[1]See III.1.

[2]We note here that $p_{k+1}(a)=(p_k)^2(a)-a$.

2. $p_n(2)=2$:

For n=2 we have $f^2(0)=p_2(a)=a^2-a$

and so $p_2(2)=4-2=2$.
Now assume $p_k(2)=2$. Hence

$$p_{k+1}=(p_k(2))^2-2=4-2=2 \text{ as needed.}$$

With this proven, we employ the Intermediate Value Theorem to obtain that there is an attractive n-cycle in $1/2<a<2$.

Now it is possible that these attractive n-cycles which we have found are degenerate. A degenerate n-cycle is also a q-cycle for q less than n (a 3-cycle is also a degenerate 9-cycle for instance). Clearly we are not satisfied at this point. If so, we would have contented ourself with noting that there is a 1-cycle at a=0 which is a degenerate n-cycle for all $n\epsilon\mathbf{N}$.

So long as n is prime the only possibility for degeneracy is that of the n-cycle being a 1-cycle.[3] But for our n-cycle (containing 0) to be a 1-cycle it must be that $f(0)=p_1(a)=-a=0$ which occurs only if a=0. But our n-cycle is in the range $1/2<a<2$.

With this we are done. We are assured, for all primes p, a non-degenerate p-cycle which is attractive. An interesting corollary is that there are an infinite number of windows in the chaotic regime.

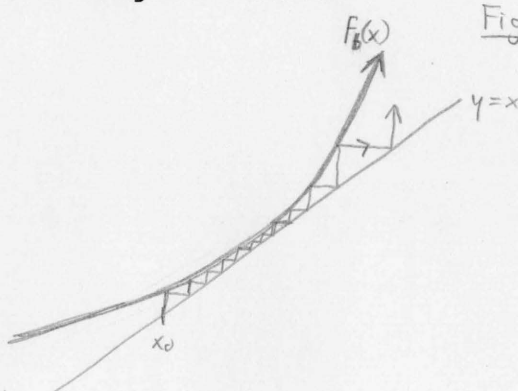
---

[3]For a proof of this see Appendix D.

5. Intermittency

Just before a window arises in the chaotic regime an odd phenomena known as "intermittency" occurs. For our family of maps $X_{n+1}=(X_n)^2-a$, for instance, the parameter "a" reaches the critical value necessary to create an attractive 3-cycle at 1.75: for "a" values just less than 1.75, (x_n) has bursts of chaotic behavior mixed with apparent convergence to a 3-cycle.[1,2] This intermittent behavior just prior to a <u>tangency</u> <u>bifurcation</u>[3] is easily explained diagrammatically. For "b" just less than $b_{critical}$ we have the situation depicted in Figure III.12.

Figure III. 12



The map $f_b^p(x)$ is nearly but not quite tangent[4] to y=x, and thus (x_n)

---

[1]Cvitanovic, 30.

[2]Bai-Lin, 45-7.

[3]A <u>tangency bifurcation</u> occurs, and a window of period n appears, when $f^n(x)$ finally becomes tangent to y=x, yielding an attractive n-cycle.

[4]Nearly tangent at n different points, only one of which is shown in Figure 1.

will act chaotic until chancing to land near q, in which case (x_n) will seem to converge to a p-cycle for a few iterations. Soon (x_n) escapes from the influence of q and returns to chaotic behavior beginning the process all over again.

The closer "b" is to $b_{critical}$, the more prolonged will be the orderly bursts.[5] On a detailed bifurcation diagram this is visible in the form of increasingly high concentrations of iterates around the n points of near-tangency as the "b" approaches a critical value. A good picture of this is found in Figure III.11 which is a segment of a detailed bifurcation diagram showing "a" near 1.75, the critical value for the 3-cycle window.

---

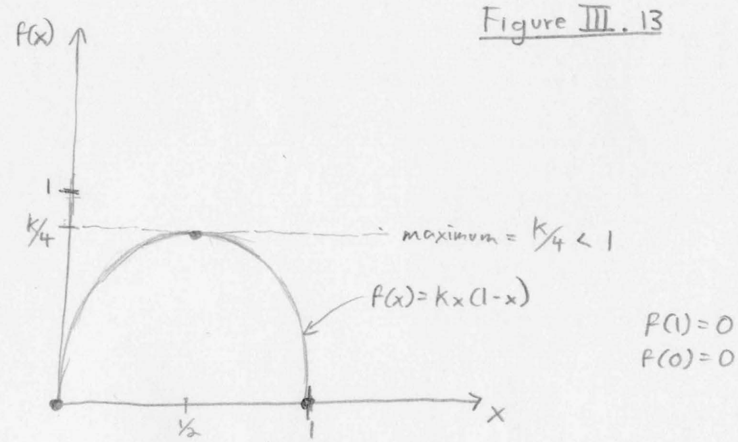
[5]Cvitanovic, 31.

6. The Chaotic Regime

For most k-values between the accumulation point 2^{∞} of the 2^n-cycles (k=3.57)[1] and k=4, the dynamical behavior of (x_n) under the logistic map is chaotic. The exceptions are the k-values that correspond to the various windows in the chaotic regime.

The chaotic behavior for various k in the chaotic regime that actually produce chaos is <u>not</u> uniform. At k=4, for instance (x_n) distributes itself evenly over the whole interval [0,1] for almost all $x_0 \epsilon [0,1]$.[2,3] For k slightly less than 4, the sequence (x_n) is still evenly distributed, but it is restricted to the interval [k(1-k/4),k/4].[4] The reason for this is seen in Figure III.13.

Figure III.13



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[1] May, 464.

[2] Kadanoff, 49.

[3] The exceptions are those isolated points which are repellant n-cycles--there is still a solution to f(x)=x in the chaotic regime, for instance, though it has long been repellant.

[4] Kadanoff, 50.

All x_n for $n>0$ are necessarily less than $k/4$ as $k/4$ is the maximum of the logistic map with parameter k. The other bound is more difficult to establish and is left unproven.

As k decreases from 4, the band constraining the chaotic (x_n) contracts but no qualitative changes occur (ignoring "windows"). At the point k_2' in Figure III.14 (next page), however, a qualitative change does occur. The random-like band splits into two branches. The sequence (x_n) now alternates between the two branches, but appears to be randomly distributed, i.e. chaotic, in each. If x_k is in the lower branch then x_{k+1} will be in the upper branch and vice versa, but this is the only apparent regularity.

At k_4' the two chaotic branches split into four, and (x_n) now visits each of these four branches in an orderly pattern, but remains chaotic inside each one. As k continues to decrease toward 3.57 [5] the number of bands continues to double, with each of these bands becoming increasingly narrow islands of chaos. This doubling process mirrors the pitchfork bifurcations (2^n-cycles) that lead up to 3.57... from the left, and in fact, the point of accumulation of the 2^n-chaotic branches is also $2^\infty \approx 3.57$. [6]

---

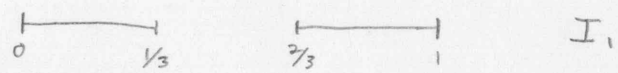
[5]The beginning of the chaotic regime is at $k \approx 3.57 = 2^\infty$.

[6]Kadanoff, 50.

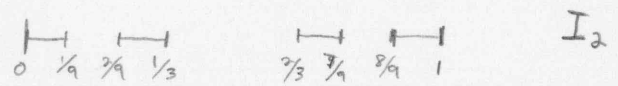
Appendix A: Fractals

Strange attractors are of fractal form. A fractal object has a peculiar geometry: it is self-similar at all scales. Any particular piece of a fractal, when viewed closely, resembles the entire structure.[1,2]

A typical fractal is the Cantor Middle Thirds set formed by the following process. Take the unit interval I_0



and remove the middle third. Call this I_1; now remove the middle third of each interval in I_1 to arrive at I_2.



The limit of I_n as n tends to infinity is the fractal Cantor Middle Thirds set.

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[1]Holden, <u>New Scientist</u>, 14.

[2]The reader is referred to the writings of Benoit Mandelbrot for further information on fractals.

Appendix B: Catastrophe Theory

Catastrophe Theory is the systematic study of discontinuities in phenomena. In the dynamical system $X_{n+1}=aX_n+b$ we established (Part II) that $|a|<1$ causes (x_n) to converge while if $|a|>1$ the sequence (x_n) diverges. Catastrophe Theory concerns itself with characterizing radical qualitative transitions (<u>"catastrophes"</u>) such as that occurring at $|a|=1$.[1,2]

Similarly there are catastrophes to be found in the bifurcation diagrams of Part III. As we "tune" the parameter k of the logistic family past 3.00, for instance, we see a abrupt transition from an attractive 1-cycle to an attractive 2-cycle: another form of "catastrophe" would be the advent of a p-window in the chaotic regime.

---

[1]Saunders, 1-13.

[2]Ekeland, 79-89.

Appendix C: Bifurcation Diagrams

The bifurcation diagram for a class of iterative equations such as $X_{n+1}=(X_n)^2-a$ shows the limiting behavior of the sequence (x_n) over a range of interesting parameter values "a".

In the case of $X_{n+1}=(X_n)^2-a$ the range of interesting dynamics is "a" between -1/4 and 2. Outside this interval the dynamics are trivial as outlined in section III.2.

Once the interval of interest is determined, the bifurcation diagram is easily generated. In the above case, for instance, a hundred or a thousand "a" values are picked at regular intervals between -1/4 and 2, and the attractor for each "a" value determined.[1] These attractors are then plotted in succession to give a picture like that in Figures III.2 and III.3. Appendix E is a Pascal program that produces such diagrams.[2]

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[1]See I.6 for this procedure.

[2]Program written by Dana Borger, NSF.

Appendix D: Degeneracy

Theorem A point x that is p-cyclic where p is prime is, if degenerate, necessarily a 1-cycle.

Proof If p=2 then we are immediately done. We proceed by contradiction for p>2. Assume x is p-cyclic and q-cyclic for q between 1 and p, but not 1-cyclic. Then since p and q are relatively prime:

$$ap-bq=1 \qquad \text{for some positive integers } a,b.$$

Hence, ap=bq+1. But this means that

$$x_{1+pa}=x_{1+p}=x_1=x_{1+q}=x_{1+qb}=x_{pa}.$$

Hence, $x_{1+pa}=x_{pa}$. But this means that x is of period one, a contradiction.

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