Norms of Composition Operators on the Hardy Space

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Chapter One: General Background

In this chapter, we will be introducing the basic concepts necessary to understand composition operators and how they work. Our composition operators will act on a Hilbert space, specifically the Hardy space. We will begin with a discussion what a Hilbert space is, and what its properties are.

Section One: Hilbert Spaces

You will recall that in Euclidean 3-space the angle between two vectors can be measured using their dot (or inner) product. In this section, we will introduce a more abstract notion of an inner product, and show how this idea can be used to add "Hilbert space" structure to a vector space. Many useful functions in the sciences have a Hilbert space as their natural domain. We will begin by defining an inner product on any complex vector space. We will also introduce the concept of the norm of a vector, which, as in the $\mathbf{R}^3$ case, can be interpreted as the length of a vector, and prove the Cauchy-Schwarz inequality for inner product spaces.

Definition 1.1: An *inner product* on a complex vector space V is a function $\varphi : V \times V \to \mathbf{C}$ such that for $f, f_1, f_2, g \in V$, $\alpha_1, \alpha_2 \in \mathbf{C}$:

1) $\varphi(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 \varphi(f_1, g) + \alpha_2 \varphi(f_2, g)$;
2) $\varphi(g, f) = \overline{\varphi(f, g)}$ (where $\overline{\varphi(f, g)}$ is the complex conjugate of $\varphi(f, g)$); and
3) $\varphi(f, f) \geq 0$, and $\varphi(f, f) = 0$ if and only if $f = 0$.

Note that from 1) and 2), one can deduce that $\varphi(g, \alpha_1 f_1 + \alpha_2 f_2) = \overline{\alpha_1}\varphi(g, f_1) + \overline{\alpha_2}\varphi(g, f_2)$ in the following manner:

$$\begin{aligned} \varphi(g, \alpha_1 f_1 + \alpha_2 f_2) &= \overline{\varphi(\alpha_1 f_1 + \alpha_2 f_2, g)} \\ &= \overline{\alpha_1 \varphi(f_1, g)} + \overline{\alpha_2 \varphi(f_2, g)} \\ &= \overline{\alpha_1}\varphi(g, f_1) + \overline{\alpha_2}\varphi(g, f_2). \end{aligned}$$

Thus, the inner product is linear in the first coordinate variable, and conjugate linear in the second. Common notation for the inner product of two elements f and g is $\langle f, g \rangle$.

Definition 1.2: An *inner product space* is a vector space with an inner product.

Definition 1.3: A *norm* on a vector space X is a function $\psi : X \to [0, \infty)$ such that:

1) $\psi(f) = 0$ if and only if $f = 0$;
2) $\psi(\lambda f) = |\lambda| \psi(f)$; and
3) $\psi(f + g) \leq \psi(f) + \psi(g)$.

The norm of a vector f is commonly denoted $\|f\|$. For $\mathbf{R}^3$, the function $\psi : \mathbf{R}^3 \to [0, \infty)$ such that $\psi(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ is a norm. The following inequality will help us see that for an inner product space X, the function $\tau : X \to [0, \infty)$ such that $\tau(f) = \sqrt{\langle f, f \rangle}$ is a norm.

Theorem 1.4: (*The Cauchy-Schwarz Inequality*): If f and g are in the inner product space X, then

$$|\langle f, g\rangle| \leq \tau(f)\tau(g),$$

where $\tau : X \to [0, \infty)$ is given by $\tau(h) = \sqrt{\langle h, h \rangle}$.

Proof: Let f and g be in an inner product space V such that $\langle f, f\rangle = \langle g, g\rangle = 1$. Choose θ such that $e^{i\theta}\langle f, g\rangle = |\langle f, g\rangle|$. Then

$$0 \leq \langle e^{i\theta} f - g, e^{i\theta} f - g\rangle = -2Re\langle e^{i\theta} f, g\rangle + 2.$$

Thus, because $Re\langle e^{i\theta} f, g\rangle = Re(e^{i\theta}\langle f, g\rangle) = |\langle f, g\rangle|$, we have

$$|\langle f, g\rangle| \leq 1 = \tau(f)\tau(g),$$

as desired. Now, let a and b be two arbitrary vectors in X. If either $a = 0$ or $b = 0$, then the inequality is trivially true (both sides are 0). If a and b are both non-zero, then

$$\langle \frac{a}{\tau(a)}, \frac{a}{\tau(a)}\rangle = \langle \frac{b}{\tau(b)}, \frac{b}{\tau(b)}\rangle = 1$$

so that

$$\begin{aligned} 1 &\geq |\langle \frac{a}{\tau(a)}, \frac{b}{\tau(b)}\rangle| \\ &= \frac{|\langle a, b\rangle|}{\tau(a)\tau(b)}, \end{aligned}$$

and thus we have,

$$|\langle a, b\rangle| \leq \tau(a)\tau(b).$$

Proposition 1.5: For any inner product space X, the function $\tau : X \to [0, \infty)$ defined by $\tau(f) = \sqrt{\langle f, f\rangle}$ is a norm.

Proof: We will show τ satisfies each of the requirements in the definition of norm.

1) For $f = 0$, $\tau(f) = \sqrt{\langle 0, 0\rangle} = 0$. If $\tau(f) = 0$, then $\langle f, f\rangle = 0$, and, by the definition of inner product, $f = 0$.
2) $\tau(\lambda f) = \sqrt{\langle \lambda f, \lambda f\rangle} = \sqrt{|\lambda|^2\langle f, f\rangle} = |\lambda|\sqrt{\langle f, f\rangle} = |\lambda|\tau(f)$
3)

$$\begin{aligned} \tau(f+g) &= \sqrt{\langle f+g, f+g\rangle} \\ &= \sqrt{\langle f, f\rangle + \langle g, g\rangle + 2Re\langle f, g\rangle} \\ &\leq \sqrt{\langle f, f\rangle + \langle g, g\rangle + 2|\langle f, g\rangle|} \\ &\leq \sqrt{\tau(f)^2 + \tau(g)^2 + 2\tau(f)\tau(g)} \\ &= \tau(f) + \tau(g) \end{aligned}$$

The last inequality follows from that of Cauchy-Schwarz.

Thus, τ is a norm and in the remaining text, when f is in an inner product space, $\|f\|$ will denote $\sqrt{\langle f, f\rangle}$. The Cauchy-Schwarz inequality can now be rewritten:

$$|\langle f, g\rangle| \leq \|f\|\|g\|.$$

Definition 1.6: A *metric* on a complex vector space X is a function $\rho : X \times X \to [0, \infty)$ such that:

1) $\rho(f, g) = 0$ if and only if $f = g$;
2) $\rho(f, g) = \rho(g, f)$; and
3) $\rho(f, h) \leq \rho(f, g) + \rho(g, h)$.

The metric measures the "distance" from one vector to another. Note that item 3) is the familiar triangle inequality, so named because in $\mathbf{R}^2$ with the usual Euclidean metric, it says that the length of the any side of a given triangle is less than the sum of the lengths of the two other sides. In a normed vector space, it is easy to verify that $\rho(f, g) = \|f - g\|$ is a metric. Thus, any norm induces a metric, and therefore any inner product induces a metric via its associated norm.

Definition 1.7: A *Hilbert space* is a complex-linear, inner-product space which is complete in the metric induced by the inner product.

Recall that *complete* means that any Cauchy sequence in the space converges to a vector in the space. A Hilbert space is thus an abstraction of R^3. The inner product of two vectors is analogous to the "dot product" or "scalar product", and the norm of any vector is simply the "distance" to the origin.

Examples 1.8: Some other common examples of a Hilbert space are:

1) The set of complex numbers $\mathbf{C}$, with an inner product:

$$\langle a + bi, c + di\rangle = (a + bi)\overline{(c + di)}.$$

2) The set $\mathbf{C}^n$ of vectors with n complex entries, , with inner product (for $(a(k))_{k=0}^{n-1} \in \mathbf{C}^n$):

$$\langle (a(k)), (b(k))\rangle = \sum_{k=0}^{n-1} a(k)\overline{b(k)}.$$

That this is an inner product is an exercise left to the interested reader.

3) The space l^2, of all sequences $(f(n))_{n=0}^{\infty}$ of complex numbers such that:

$$\sum_{n=0}^{\infty} |f(n)|^2 < \infty,$$

with inner product:

$$\langle (a(n)), (b(n)) \rangle = \sum_{n=0}^{\infty} a(n)\overline{b(n)},$$

is a Hilbert space, as shown below.

4) For $\mathbf{U}$ the open unit disk centered at the origin, the Hardy space, denoted H^2, is the set of all complex analytic functions on $\mathbf{U}$ whose Taylor expansions about zero have coefficients that are l^2 sequences. Thus an analytic function $f : \mathbf{U} \to \mathbf{C}$ given by $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ is in H^2 provided $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$ (here we have used $\hat{f}(n)$ to denote the n^{th} taylor coefficent of f in its expansion about zero, so that $\hat{f}(n) = \frac{f^{(n)}(0)}{n!}$). The inner product of f and g in H^2 is:

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.$$

As the reader has surely noticed, H^2 and l^2 are essentially the same Hilbert space: the mapping $\sum_{n=0}^{\infty} \hat{f}(n) z^n \to (\hat{f}(n))$ is a vector space isomorphism of H^2 onto l^2 that preserves the inner product.

5) From quantum physics, the space of the wave functions for all states of a given particle with a given energy is an inner product space. Clearly it is a vector space. With inner product $\langle \varphi_k, \varphi_{k'} \rangle = \int_{-\infty}^{\infty} \varphi_k(x)\overline{\varphi_{k'}(x)}dx$, this space meets the requirements for a Hilbert space.

Theorem 1.9: The space l^2 is a Hilbert space.

Proof : Note that l^2 is a complex-linear vector space because if f and g are in l^2, then $|(f+g)(n)|^2 \leq 2|f(n)|^2 + 2|g(n)|^2$ for each n, and thus $f+g$ is an l^2 sequence.

Now we shall demonstrate that l^2 has the other properties of a Hilbert space. Let N be a positive integer. Define F_N and G_N for $f, g \in l^2$ as $F_N = (|f(0)|, |f(1)|, ..., |f(N)|)$, and $G_N = (|g(0)|, |g(1)|, ..., |g(N)|)$. Note that these are elements of $\mathbf{C}^{N+1}$. Then, using the inner product for $\mathbf{C}^{N+1}$ defined above, as well as the Cauchy-Schwarz inequality, we see that:

$$\begin{aligned}\sum_{n=0}^{N} |f(n)\overline{g(n)}| &= |\langle F_N, G_N \rangle| \\ &\leq \|F_N\| \|G_N\| \\ &= \Big(\sum_{n=0}^{N} |f(n)|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{N} |g(n)|^2\Big)^{\frac{1}{2}} \\ &\leq \|f\|_{l^2} \|g\|_{l^2}.\end{aligned}$$

Thus, the series $\sum_{n=0}^{\infty} f(n)\overline{g(n)}$ converges absolutely, and therefore φ defined on $l^2 \times l^2$ by $\varphi(a(n), b(n)) = \sum a_n \overline{b_n}$ is a well defined function, which takes on values in $\mathbf{C}$. Showing that the φ satisfies the definition of an inner product follows readily, and is left to the reader as an exercise.

Now we must establish that l^2 is complete. Suppose $(f_k)_{k=0}^{\infty}$ is a Cauchy sequence in l^2, meaning that for each $\epsilon > 0$, there exists a non-negative integer m such that for all $i > j > m$, $\|f_i - f_j\| < \epsilon$. Then, for each non-negative integer n, we have

$$|f_i(n) - f_j(n)| \leq \|f_i - f_j\|_{l^2},$$

so that $(f_k(n))_{k=0}^{\infty}$ is a Cauchy sequence in $\mathbf{C}$ for all n. Since $\mathbf{C}$ is complete, $(f_k(n))_{k=0}^{\infty}$ converges, and we may define the function F from the non-negative integers to $\mathbf{C}$ by $F(n) = \lim_{k\to\infty} f_k(n)$. We now claim that $\lim(f_k) = F$. We must show two things about F: first that F is in l^2, and second that $\lim_{k\to\infty} \|F - f_k\|_{l^2} = 0$. As $(f_k)_{k=0}^{\infty}$ is a Cauchy sequence, it is bounded in the l^2 norm; that is, there exists a real number M such that for all k, $\|f_k\|_{l^2} \leq M$. Let P be a non-negative integer; for any integer K,

$$\begin{aligned}\sqrt{\sum_{n=0}^{P} |F(n)|^2} &\leq \sqrt{\sum_{n=0}^{P} |F(n) - f_K(n)|^2} + \sqrt{\sum_{n=0}^{P} |f_K(n)|^2} \\ &\leq \sqrt{\sum_{n=0}^{P} |F(n) - f_K(n)|^2} + M.\end{aligned}$$

Because $\lim_{K\to\infty} \sum_{n=0}^{P} |F(n) - f_K(n)|^2 = 0$ (by definition of F), F is in l^2 (because $\sum_{n=0}^{P} |F(n)|^2 \leq M$ for all P).

We will now show that f_K converges to F. Given $\epsilon > 0$, choose a non-negative real number R such that $k, j \geq R$ implies $\|f_k - f_j\| \leq \epsilon$. Then for $k \geq R$, and any Q in the non-negative integers, we have

$$\sum_{n=0}^{Q} |F(n) - f_k(n)|^2 = \lim_{j\to\infty} \sum_{n=0}^{Q} |f_j(n) - f_k(n)|^2 \leq \epsilon^2,$$

and the inequality holds because for every $j \geq R$, $\sum_{n=0}^{Q} |f_j(n) - f_k(n)|^2 \leq \|f_j(n) - f_k(n)\|^2 \leq \epsilon^2$. Then, since Q is arbitrary, $\|F - f_k\|_{l^2} \leq \epsilon$ for any $k \geq R$, and therefore l^2 is complete.

Definition 1.10: An *orthonormal basis* of a Hilbert space H is an orthonormal subset of H such that H is the closed linear span of the set.

Recall the set $\{f_\alpha : \alpha \in A\}$ of vectors in H is orthonormal provided $\|f_\alpha\| = 1$ for all $\alpha \in A$, and $\langle f_\beta, f\gamma \rangle = 0$ for $\beta, \gamma \in A$, and $\beta \neq \gamma$. Also recall that the linear span of a set is the set of all finite linear combinations of the elements of the set. Thus a vector $g \in H$ is in the closed linear span of $\{f_\alpha : \alpha \in A\}$ if and only if $g = \lim_{n\to\infty} \sum_{k=0}^{n} a_k f_{\alpha_k}$, where $\alpha_k \in A$ and $a_k \in \mathbf{C}$ for all integers k.

Definition 1.11: The *dimension* of a Hilbert space H, denoted dim H, is the cardinality of any orthonormal basis of H. (It can be shown that any two orthonormal bases of a Hilbert space have the same cardinality [3, Theorem 3.30].)

Recall that two sets have the same cardinality if there is a bijection between them. Note that dim $\mathbf{C}^n$ is n, and dim l^2 is $\aleph_0$, the cardinality of the natural numbers. (For example, a basis of l^2 is $\{(1,0,0,...),(0,1,0,0,...),...\}$.)

Recall also from linear algebra that, given any closed subspace M of the vector space $\mathbf{R}^n$, any element of $\mathbf{R}^n$ can be expressed as the sum of a vector in M and a vector orthogonal to everything in M. The same is true for Hilbert spaces.

Theorem 1.12: If M is a closed subspace of the Hilbert space H, and f is a vector in H, then there exist unique vectors g in M, and h in the set of vectors perpendicular to M such that $f = g + h$.

The proof of this theorem can be found in *Banach Algebra Techniques in Operator Theory* by R. G. Douglas [3, Theorem 3.21].

Section Two: The Hardy Space

In the previous section, we defined the Hardy space H^2. In this chapter we will explore some of its properties in preparation for our study of composition operators.

The inner product on the Hardy space, as discussed in the last section, is

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)},$$

where $\hat{f}(n)$ is the n^{th} Taylor coefficient of f for the series centered at zero. The norm induced by this inner product is

$$\|f\| = \Big(\sum_{n=0}^{\infty} |\hat{f}(n)|^2\Big)^{\frac{1}{2}}.$$

Examples 1.13: The following are some examples of elements of the Hardy space.

1) $f(z) = z^2$, or any polynomial in z, is clearly in H^2.
2) $f(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ is in H^2, as $(\frac{1}{n!})$ is an l^2 sequence.
3) Any bounded, analytic function on **U** is an element of the Hardy space. (See Theorem 1.15 below.)
4) $\text{Log}(1-z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}z^n$ is in H^2, even though it is not bounded on **U**, as $(\frac{1}{n})$ is an l^2 sequence.
5) Any function of the form $f(z) = (1-z)^{-p}$, is in H^2, for $0 \leq p < \frac{1}{2}$. (See Theorem 1.17 below.)

Another method of computing the norm of a function in the Hardy space is presented in the following theorem.

Theorem 1.14: For f analytic on **U**,

$$\sum_{n=0}^{\infty} |\hat{f}(n)|^2 = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

Proof: First, note that if both sides of the equation are infinite, the equality holds. Now let $r \in \mathbf{R}$ be such that $0 < r < 1$. If $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$, then:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |\sum_{k=0}^{n} \hat{f}(k)(re^{i\theta})^k|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{n} \hat{f}(k) r^k e^{ik\theta} \sum_{k=0}^{n} \overline{\hat{f}(k)}(r^k e^{-ik\theta}) d\theta \\ &= \sum_{k=0}^{n} r^{2k} |\hat{f}(k)|^2 \\ &\leq \sum_{n=0}^{\infty} |\hat{f}(n)|^2.\end{aligned}$$

Because the power series representing f converges uniformly on the circle of radius r, it follows upon letting $n \to \infty$ that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{k=0}^{\infty} r^{2k}|\hat{f}(k)|^2$$
$$\leq \sum_{n=0}^{\infty} |\hat{f}(n)|^2.$$

The equality in the expression above shows

$$r \mapsto \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$$

is increasing in r. Thus, $\lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta$ exists and is less than or equal to $\sum_{n=0}^{\infty}|\hat{f}(n)|^2$.

If $\lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta$ is finite, then $\lim_{r\to 1^-}\sum_{k=0}^{\infty} r^{2k}|\hat{f}(k)|^2$ is finite. (Note we did not use finiteness of $\sum_{n=0}^{\infty}|\hat{f}(n)|^2$ to prove $\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta = \sum_{k=0}^{\infty} r^{2k}|\hat{f}(k)|^2$.) Fix $n \in \mathbf{Z}$, then

$$\sum_{k=0}^{n}|\hat{f}(k)|^2 = \lim_{r\to 1^-}\sum_{k=0}^{n} r^{2k}|\hat{f}(k)|^2$$
$$\leq \lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta.$$

Thus, because n is arbitrary, $\lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta \geq \sum_{k=0}^{\infty}|\hat{f}(k)|^2$, and the theorem follows.

This theorem can be readily used to prove items 3) and 5) above.

Theorem 1.15: Any bounded, analytic function on $\mathbf{U}$ is an element of the Hardy space.

Proof: Let g be an analytic function on $\mathbf{U}$ such that for all $z \in \mathbf{U}$, $|g(z)| \leq M$ for some $M \in \mathbf{R}$. Then

$$\lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}|g(re^{i\theta})|^2 d\theta \leq \lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi} M^2 d\theta = M^2,$$

and thus $\sum_{k=0}^{\infty}|\hat{g}(k)|^2 \leq M^2 < \infty$ and g is in H^2.

Before proving the second of these two items, we will need the following lemma.

Lemma 1.16: For θ in the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$, $|\sin\theta| > |\frac{\theta}{2}|$.

Proof: By the mean–value theorem, there exists c between 0 and θ such that $|\sin\theta - \sin 0| = |\theta\cos c| > |\frac{\theta}{2}|$. Thus, $|\sin\theta| > |\frac{\theta}{2}|$.

Theorem 1.17: Any function of the form $h(z) = (1-z)^{-p}$, is in H^2, for $p < \frac{1}{2}$.

Proof: Let h and p be as above. Then,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^{2p}}d\theta \leq \frac{1}{2\pi}\int_{\frac{7\pi}{4}}^{\frac{\pi}{4}} \frac{1}{|\mathrm{Im}(1-re^{i\theta})|^{2p}}d\theta + \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} \frac{1}{|1-re^{i\theta}|^{2p}}d\theta$$
$$\leq \frac{1}{2\pi}\left(\int_{\frac{7\pi}{4}}^{\frac{\pi}{4}} |(r\sin\theta)^{-p}|^2 d\theta + \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} |\frac{1}{\sqrt{2}}|^{-2p} d\theta\right)$$
$$\leq 3\pi 2^{p-1} + \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} \frac{1}{|r\sin\theta|^{2p}}d\theta$$
$$\leq 3\pi 2^{p-1} + \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} \frac{1}{|r\theta/2|^{2p}}d\theta,$$

which is clearly finite because $2p < 1$. Thus, $h(z) = (1-z)^{-p}$ is in H^2.

Theorem 1.18: For any $\alpha \in \mathbf{U}$, there exists an element k_α of the Hardy space, which, for any f in H^2 has the following property:

$$\langle f, k_\alpha \rangle = f(\alpha).$$

Proof: Let α be in $\mathbf{U}$, and let k_α be defined on $\mathbf{U}$ as follows:

$$k_\alpha(z) = \frac{1}{1-\overline{\alpha}z}.$$

Note $k_\alpha = \sum_{k=0}^{\infty}(\overline{\alpha}z)^k$ is in H^2, as α is in $\mathbf{U}$. Clearly $\langle f, k_\alpha \rangle = \sum_{k=0}^{\infty} \hat{f}(k)\alpha^k = f(\alpha)$.

The function k_α is called the *reproducing kernel* for H^2 at α, and can be used to obtain a growth estimate for any element of the Hardy space.

Corollary 1.19: For $f \in H^2$ and $\alpha \in \mathbf{C}$,

$$|f(\alpha)| \leq \frac{\|f\|}{(1-|\alpha|^2)^{\frac{1}{2}}}.$$

Proof: Using the Cauchy-Schwarz equality and k_α as defined above:

$$|f(\alpha)| = |\langle f, k_\alpha \rangle| \leq \|f\|\|k_\alpha\| = \frac{\|f\|}{(1-|\alpha|^2)^{\frac{1}{2}}}.$$

Section Three: Linear Operators

Definition 1.20: For vector spaces X and Y over the field F, a *linear operator* from X to Y is a function $\Lambda : X \to Y$ such that $\Lambda(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \Lambda(x_1) + \alpha_2 \Lambda(x_2)$ for all $\alpha_1, \alpha_2 \in F$ and $x_1, x_2 \in X$.

Recall that, for all linear functions S, $S(0) = 0$. We will say an operator from a vector space X to X is an *operator on* X.

Examples 1.21: The following are some examples of linear operators on vector spaces:

1) Multiplication by a 3×3 matrix is a linear operation on $\mathbf{R}^3$.
2) The identity operator, I, is linear on any vector space.
3) For linear operators S and $T : X \to Y$, any linear combination of S and T is a linear operator.
4) In $\mathbf{R}^3$, rotation about any axis through 0, by say 30°, is a linear operation. (Of course this operation is equivalent to multiplication by a 3×3 matrix, and is represented in example 1).)
5) For any sequence $a = (a(0), a(1), ...)$ in l^2, let $V : l^2 \to l^2$ be defined by $V(a) = (a(1), a(2), a(3), ...)$. (That is, remove the first entry in $(a(n))_{n=0}^{\infty}$.) This operator is commonly called the *backward shift* operator. On H^2, $Vg = \frac{g(z)-g(0)}{z}$ for all $g \in H^2$.
6) Let $U : l^2 \to l^2$ be defined by $U(a) = (0, a(0), a(1), ...)$. This operator is commonly called the *forward shift* operator, and on H^2, $Uf = zf(z)$.
7) Let $\mathcal{H}(\mathbf{U})$ be the set of holomorphic functions on the open unit disk, and let $\varphi \in \mathcal{H}(\mathbf{U})$ be such that the range of φ is contained in $\mathbf{U}$. Then the *composition operator* on $\mathcal{H}(\mathbf{U})$ with symbol φ, denoted C_φ is defined by $C_\varphi h = h \circ \varphi$. That C_φ is a linear operator is easy to verify.
8) For $(\alpha_k)_{k=0}^{\infty}$ bounded, the operator M_α on l^2 defined by $M_\alpha a \to (\alpha_k a(k))$ is linear for all $a \in l^2$.
9) Let H be a Hilbert space. Then the operator $\Lambda_g : H \to \mathbf{C}$ defined by $\Lambda_g(f) = \langle f, g \rangle$ for all $f \in H$ is a linear operator.

In the list above, we have defined several operators on Hilbert spaces. Now we will discuss some properties of these operators, and will prove some helpful theorems regarding them. Note that example 7), the composition operator, will be analyzed extensively in the next chapter.

Definition 1.22: An operator S from the Hilbert space X to the Hilbert space Y is *bounded* if there exists a positive real number M such that, for all $x \in X$, $\|Sx\| \leq M\|x\|$. In this case, we would say *S is bounded by M*.

Theorem 1.23: Let $S : X \to Y$ be a linear operator for Hilbert spaces X and Y. Then the following are equivalent:

1) S is continuous.

2) S is continuous at zero.

3) S is bounded.

Proof: Clearly 1) implies 2). Assume S is continuous at zero. Then there exists $\delta > 0$ such that $\|f\| < \delta$ implies $\|Sf\| < 1$. Thus, for any nonzero $g \in X$, we have

$$\|Sg\| = \frac{2\|g\|}{\delta}\|S(\frac{\delta}{2\|g\|}g)\| < \frac{2}{\delta}\|g\|,$$

and thus S is bounded.

Now assume S is bounded by M. Let $\epsilon > 0$. Then, for $f_1, f_2 \in X$ such that $\|f_1 - f_2\| \leq \frac{\epsilon}{M}$,

$$\begin{aligned}\|Sf_1 - Sf_2\| &= \|S(f_1 - f_2)\| \\ &\leq M\|f_1 - f_2\| \\ &= \epsilon,\end{aligned}$$

and S is continuous.

Definition 1.24: The *norm* of a bounded linear operator $S : X \to Y$, denoted by $\|S\|$, is given by $\|S\| = \sup\{\frac{\|Sx\|}{\|x\|} : x \in X \setminus \{0\}\}$.

Note that $\|S\| = \min\{M : \|Sx\| \leq M\|x\| \text{ for all } x \in X\}$; in particular, $\|Sx\| \leq \|S\|\|x\|$ for all $x \in X$. Also notice $\|S\| = \sup\{\|Sx\| : \|x\| = 1\}$.

The notation for the operator norm and the norm of an element of a Hilbert space are the same, and should not be confused. The meaning of the norm notation should be clear from the context. Thus, the norms of the operators in examples 1.21 5) and 6) are 1, and the norm of the operator in 8) is $\sup\{|\alpha_k| : k \in 0, 1, 2, 3, ...\}$.

Proposition 1.25: Suppose T and S are bounded linear operators on a Hilbert space H. Then TS is a bounded linear operator, and $\|TS\| \leq \|T\|\|S\|$.

Proof: First, we observe that $TS(\alpha f + \beta g) = T(\alpha Sf + \beta Sg) = \alpha TSf + \beta TSg$, and thus TS is linear. By definition, $\|TS\| = \sup\{\frac{\|TSf\|}{\|f\|} : f \in H,\ f \neq 0\}$. From the definition of norm, we see $\|TSf\| \leq \|T\|\|Sf\| \leq \|T\|\|S\|\|f\|$, and thus $\frac{\|TSf\|}{\|f\|} \leq \|T\|\|S\|$, and $\|TS\| \leq \|T\|\|S\|$.

The following proposition identifies the norm of the operator in example 9) above.

Proposition 1.26: The operator norm of Λ_g, as defined by $\Lambda_g(f) = \langle f, g\rangle$ for all $f \in H$ is $\|g\|$.

Proof: For $g = 0$, $\|\Lambda_g\| = \|g\| = 0$. Suppose $g \neq 0$. Let $B = \{\frac{|\langle f,g\rangle|}{\|f\|} : f \neq 0\}$. Then $\|\Lambda_g\| = \sup B$. By the Cauchy-Schwarz inequality,

$$\frac{|\langle f,g\rangle|}{\|f\|} \leq \frac{\|f\|\|g\|}{\|f\|} = \|g\|.$$

Thus $\|g\|$ is an upper bound of B, and $\|g\| \geq \|\Lambda_g\|$. Now use the fact that f can take on any non-zero value in H (in particular g), and $\frac{|\langle f,g\rangle|}{\|f\|} = \frac{\|g\|^2}{\|g\|} = \|g\|$. Thus, $\|g\| \in B$, and $\|\Lambda_g\| \geq \|g\|$.

Definition 1.27: A *bounded linear functional* on a Hilbert space H is a bounded, linear operator from H to the complex plane.

In Example 1.21 number 9), we presented for any Hilbert space a class of bounded linear functionals. Now we show that all bounded linear functionals on any given Hilbert space belong to that class.

Theorem 1.28 (The Riesz Representation Theorem): Let H be a Hilbert space, and S a bounded linear functional with domain H. Then there exists a unique $g \in H$ such that for any $f \in H$, S can be represented in the following form:

$$Sf = \langle f,g\rangle.$$

Proof: Let $\mathcal{K} = \ker\ S$ $(= \{f \in H : Sf = 0\})$. As S is continuous, $\mathcal{K}$ is a closed subspace of H. If $\mathcal{K} = H$, then $Sf = \langle f, 0\rangle$, and we are done. For $\mathcal{K} \neq H$, there exists $h \in H$ such that $\|h\| = 1$ and h is orthogonal to all $k \in \mathcal{K}$ (i.e. $\langle h,k\rangle = 0$) by Theorem 1.12. Then, as $h \notin \mathcal{K}$, $Sh \neq 0$. Now, for $f \in H$, $(f - (\frac{Sf}{Sh})h) \in \mathcal{K}$, as $S(f - (\frac{Sf}{Sh})h) = 0$. Therefore, since h is orthogonal to $\mathcal{K}$,

$$\begin{aligned} 0 &= \langle f - \frac{Sf}{Sh}h, h\rangle \\ &= \langle f,h\rangle - \frac{Sf}{Sh} \end{aligned}$$

or, for all $f \in H$, $Sf = \langle f, \overline{Sh}h\rangle$ and thus, $Sf = \langle f,g\rangle$, for $g = \overline{Sh}h$.

Now we shall prove that this g is unique. If $\langle f, g_1\rangle = \langle f, g_2\rangle$ for all $f \in H$, then for $f = g_1 - g_2$, subtracting gives $\langle g_1 - g_2, g_1 - g_2\rangle = 0$, and $g_1 = g_2$.

Thus, $Sf = \langle f,g\rangle$, for unique $g \in H$.

For the remainder of this paper, we will be concerned with bounded linear operators on Hilbert spaces (focusing primarily composition operators). Any such operator T has a natural "companion", or "dual" operator T^*, called the adjoint of T. The existence of this dual operator is established in following theorem.

Theorem 1.29: For any bounded linear operator T on a Hilbert space H there exists a unique bounded linear operator S such that for any $f, g \in H$ $\langle Tf,g\rangle = \langle f, Sg\rangle$.

Proof: For a fixed g in a Hilbert space H, let φ be defined by $\varphi(f) = \langle Tf, g\rangle$ for $f \in H$. It is easy to show that φ is a bounded linear functional on H, and hence there exists by the Riesz representation theorem a unique $h \in H$ such that $\varphi(f) = \langle f, h\rangle$ for all $f \in H$. Let $Sg = h$.

By definition of S, $\langle Tf, g\rangle = \langle f, Sg\rangle$ for $f, g \in H$. It is easy to verify that S is linear. Setting $f = Sg$ we obtain:

$$\|Sg\|^2 = |\langle Sg, Sg\rangle| = |\langle TSg, g\rangle| \leq \|T\|\|Sg\|\|g\|$$

for $g \in H$. Thus $\|S\| \leq \|T\|$, and S is a bounded operator on H.

To show that S is unique, suppose S_1 is another operator such that $\langle f, S_1 g\rangle = \langle Tf, g\rangle$ for $f, g \in H$. Then $\langle f, Sg - S_1 g\rangle = 0$ for $f \in H$, and setting $f = Sg - S_1 g$ we see that $Sg - S_1 g = 0$. Hence $S = S_1$ and the proof is complete.

Definition 1.30: For an operator T on a Hilbert space H, the *adjoint* of T, denoted T^* is the unique operator on H such that for $f, g \in H$, $\langle Tf, g\rangle = \langle f, T^* g\rangle$.

The reader may verify that the adjoint of matrix multiplication is multiplication by the transpose.

Proposition 1.31: For U the forward shift operator and V the backward shift operator on the Hardy space H^2, $U^* = V$.

Proof: Let $f, g \in H^2$. Then

$$\begin{aligned}\langle f, U^* g\rangle &= \langle Uf, g\rangle \\ &= \sum_{k=0}^{\infty} \hat{f}(k)\overline{\hat{g}(k+1)} \\ &= \langle f, Vg\rangle.\end{aligned}$$

Hence, $\langle f, U^* g - Vg\rangle = 0$ for all f, which implies $U^* g = Vg$.

Proposition 1.32: For a bounded linear operator S on a Hilbert space H, $\|S\| = \sup\{|\langle Sf, g\rangle| : \|f\| = \|g\| = 1\}$.

Proof: Let f and g be such that $\|f\| = \|g\| = 1$, and let $\{|\langle Sf, g\rangle| : \|f\| = \|g\| = 1\} = L$. We have

$$\begin{aligned}|\langle Sf, g\rangle| &\leq \|Sf\|\|g\| \\ &= \|Sf\| \\ &\leq \|S\|\|f\| \\ &= \|S\|,\end{aligned}$$

and thus $\|S\| \geq \sup L$. Also, given $h \in H$ such that $\|h\| = 1$, $\langle Sh, \frac{Sh}{\|Sh\|}\rangle = \|Sh\|$, and sup $L \geq \sup\{\|Sh\| : \|h\| = 1\} = \|S\|$, and the proposition is proven.

Proposition 1.33: For T a bounded linear functional on a Hilbert space H, $\|T\| = \|T^*\|$.

Proof: Observe that for f and g in H, $|\langle f, T^*g\rangle| = |\langle Tf, g\rangle| = |\langle g, Tf\rangle|$, and thus $\|T\| = \|T^*\|$ by the proposition above.

Lemma 1.34: Suppose that T is a bounded linear operator on a Hilbert space H, and $\|I - T\| < 1$. Then T is invertible, and

$$\|T^{-1}\| \leq \frac{1}{1 - \|I - T\|}.$$

Proof: If we set $\eta = \|I - T\| < 1$, then for $N \geq M$, we have

$$\begin{aligned}
\|\sum_{n=0}^{N}(I-T)^n - \sum_{n=0}^{M}(I-T)^n\| &= \|\sum_{n=M+1}^{N}(I-T)^n\| \\
&\leq \sum_{n=M+1}^{N}\|I-T\|^n \\
&= \sum_{n=M+1}^{N}\eta^n \\
&\leq \frac{\eta^{M+1}}{1-\eta},
\end{aligned}$$

and the sequence of partial sums $\{\sum_{n=0}^{N}(I-T)^n\}_{N=0}^{\infty}$ is Cauchy. If $S = \sum_{n=0}^{\infty}(I-T)^n$, then

$$\begin{aligned}
TS &= [I-(I-T)]\left(\sum_{n=0}^{\infty}(I-T)^n\right) \\
&= \lim_{N\to\infty}\left([I-(I-T)]\sum_{n=0}^{N}(I-T)^n\right) \\
&= \lim_{N\to\infty}[I-(I-T)^{N+1}] \\
&= I,
\end{aligned}$$

since $\lim_{N\to\infty}\|(I-T)^{N+1}\| = 0$. Similarly, $TS = I$, so that T is invertible, with $T^{-1} = S$. Further,

$$\|S\| = \lim_{N\to\infty}\|\sum_{n=0}^{N}(I-T)^n\| \leq \lim_{N\to\infty}\sum_{n=0}^{N}\|I-T\|^n = \frac{1}{1-\|I-T\|}.$$

Definition 1.35: For an operator $S : H \to H$, where H is a Hilbert space, the *spectrum* of S, denoted $\sigma(S)$, is defined by $\sigma(S) = \{\alpha \in \mathbf{C} : S - \alpha I \text{ is not invertible in } H\}$.

The reader should observe that the set of eigenvalues of a linear operator is a subset of its spectrum. We will now examine some examples of spectra of linear operators.

Examples 1.36:

1) For $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by multiplication by a 3×3 matrix, $\sigma(T)$ is the set of eigenvalues of T.
2) For $U : H^2 \to H^2$ the forward shift operator, $\sigma(U) = \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$. This is proven below.
3) For $V : H^2 \to H^2$ the backward shift operator, $\sigma(V) = \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$. This follows easily from 1) above, and the fact that $\sigma(T) = \overline{\sigma(T^*)}$, where the bar denotes the complex conjugate of all the elements of the set (the proof of which is left to the interested reader).

We will show below that spectra of operators on infinite dimensional Hilbert spaces consist of more than just eigenvalues. First we will prove some more general properties of the spectrum.

Theorem 1.37: For an operator $S : H \to H$ where H is a Hilbert space, $\sigma(S)$ is closed and bounded.

Proof: Let $\lambda \in \mathbf{C} \setminus \sigma(S)$, so that $S - \lambda I$ is invertible. We will show $\mathcal{B} = \{\alpha : |\alpha - \lambda| < \frac{1}{\|(S-\lambda I)^{-1}\|}\} \subset \mathbf{C} \setminus \sigma(S)$ is open. Let $\alpha \in \mathcal{B}$. Then

$$\begin{aligned}\|(S - \lambda I)^{-1}(S - \alpha I) - I\| &= \|(S - \lambda I)^{-1}(S - \alpha I - (S - \lambda I))\| \\ &\leq \|S - \lambda I\|^{-1}\|(\lambda - \alpha)I\| \\ &\leq |\lambda - \alpha|\|S - \lambda I\|^{-1} \\ &< 1.\end{aligned}$$

Thus, $(S - \lambda I)^{-1}(S - \alpha I)$ is invertible, and therefore $S - \alpha I$ is invertible, and $\mathbf{C} \setminus \sigma(S)$ is open.

For boundedness, we will show $\sigma(S) \subset \{z \in \mathbf{C} : |z| \leq \|S\|\}$. If $|\lambda| > \|S\|$, then

$$1 > \frac{\|S\|}{|\lambda|} = \|\frac{S}{\lambda}\| = \|I - (I - \frac{S}{\lambda})\|,$$

so that $I - \frac{S}{\lambda}$ is invertible by the above lemma. Therefore, $\lambda \in \mathbf{C} \setminus \sigma(S)$ (as $S - \lambda I = \lambda(1 - \frac{S}{\lambda})$), and $\sigma(S)$ is bounded.

The following theorem is proved in *Banach Algebra Techniques in Operator Theory* by R. G. Douglas [3, 2.29].

Theorem 1.38: For an operator $S : H \to H$ where H is a Hilbert space, $\sigma(S)$ is non-empty.

Thus, by the preceding theorems, $\sigma(S)$ is a non-empty, compact subset of $\mathbf{C}$.

Definition 1.39: The *spectral radius* of an operator T on a Hilbert space H, denoted $r(T)$, is a real number equal to $\sup\{|\lambda| : \lambda \in \sigma(T)\}$.

Observe that as a corollary of the preceeding definition and theorem, $r(T) \leq \|T\|$.

Proposition 1.40: For $U : H^2 \to H^2$ the forward shift operator, $\sigma(U) = \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$.

Proof: First, we will show $U - \lambda I$ is not surjective for $\lambda \in \mathbf{U}$. For 1 to be in the range of $U - \lambda I$, $g(z) = \frac{1}{z-\lambda}$ must be in H^2, which it is not, as it is not analytic on the disk. Thus, $U - \lambda I$ is not onto, and $\lambda \in \mathbf{U}$ is in the spectrum of U. As the spectrum is closed, all λ such that $|\lambda| \leq 1$ must be in $\sigma(U)$. Note that $\lambda \in \mathbf{C}$ such that $|\lambda| > 1$ cannot be in the spectrum, because $\|U\| = 1$, and $r(U) \leq \|U\|$.

We remark that U has no eigenvalues. Suppose λ is an eigenvalue of U. Then

$$(Uf)(z) = \lambda f(z)$$

for some $f \neq 0$ in H^2. Hence,

$$f(z)(z - \lambda) = 0,$$

so that $f(z) = 0$, except at λ. This is a contradiction, because f must be continuous (it is, after all analytic).

Chapter Two: Composition Operators

In this chapter, we will discuss composition operators on the Hardy space, focusing primarily on the computation of norms of these operators.

Section One: Introduction

In this section, we will introduce some basic facts about composition operators, and begin our analysis of bounds on the norms of these operators. We will begin by discussing possible forms of compostion operators.

Proposition 2.1: Let $\varphi : \mathbf{U} \to \mathbf{U}$ be analytic. Then $f \circ \varphi \in \mathcal{H}(\mathbf{U})$.

Proof: Chain rule.

Thus, for $\varphi : \mathbf{U} \to \mathbf{U}$ analytic, the operator C_φ on $\mathcal{H}(\mathbf{U})$ defined by $C_\varphi f = f \circ \varphi$ maps $\mathcal{H}(\mathbf{U})$ into $\mathcal{H}(\mathbf{U})$. Recall that C_φ is the compostion operator on $\mathcal{H}(\mathbf{U})$ with symbol φ.

Examples 2.2: The following are several examples of functions from the disk into a subset of the disk. Thus, any of these functions could be the symbol of a composition operator.

1) $\varphi(z) = \alpha$ for $\alpha \in \mathbf{U}$.
2) $\varphi(z) = \alpha z$ for $\alpha \in \mathbf{U}$.
3) Any polynomial in $\mathbf{U}$ which takes the unit disk into itself, e.g. $\varphi(z) = \frac{1}{4}z^2 + \frac{3}{4}z$.
4) Linear fractional functions of the form $\frac{p-z}{1-\bar{p}z}$ for $p \in \mathbf{U}$.
5) Let $f \in \mathcal{H}(\mathbf{U})$ be bounded and non-constant. Then for $M = \sup\{|f(z)| : z \in \mathbf{C}\}$, $\varphi(z) = \frac{f(z)}{M}$ is a symbol of a compostion operator.

Definition 2.3: A bounded linear operator T is said to be a *contraction* if for each $f \in \mathcal{H}(\mathbf{U})$, $\|Tf\| \leq \|f\|$. Note this is equivalent to $\|T\| \leq 1$.

The following theorem shows that the Hardy space H^2, a subset of $\mathcal{H}(\mathbf{U})$, is preserved under C_φ. The proof presented here is an adaptation of that which appears in *Composition Operators and Classical Function Theory* by Joel H. Shapiro [4, Section 1.3].

Theorem 2.4 (Littlewood's Theorem): Suppose $\varphi \in \mathcal{H}(\mathbf{U})$ has range contained in $\mathbf{U}$ and satisfies $\varphi(0) = 0$. Then $C_\varphi(H^2) \subseteq H^2$. Moreover, for each $f \in H^2$, $\|f \circ \varphi\| \leq \|f\|$.

Proof: Let $V : H^2 \to H^2$ (the backward shift operator) be such that for f in H^2, $Vf(z) = \sum_{n=0}^{\infty} \hat{f}(n+1)z^n$. Note that $f(z) = f(0) + zVf(z)$, and $V^n f(0) = \hat{f}(n)$. Now suppose f is a polynomial. Then $f \circ \varphi$ is bounded on $\mathbf{U}$, so that $f \circ \varphi \in H^2$. Now,

$$f(\varphi(z)) = f(0) + \varphi(z)(Vf)(\varphi(z))$$

or,

$$C_\varphi f = f(0) + M_\varphi C_\varphi V f,$$

for M_φ the multiplication operator (that is multiplication by φ). As $\varphi(0) = 0$, all terms of the power series for φ have a factor of z, and therefore $M_\varphi C_\varphi V f$ is orthogonal in H^2 to $f(0)$. Thus,

$$\begin{aligned} \|C_\varphi f\| &= |f(0)|^2 + \|M_\varphi C_\varphi V f\|^2 \\ &\leq |f(0)|^2 + \|C_\varphi V f\|^2. \end{aligned}$$

This is true because M_φ is contractive on H^2 (which can be readily proven using the integral definition of norm). Now substitute $Vf, V^2 f, V^3 f, \ldots$ for f in the above equations, and

$$\begin{aligned} \|C_\varphi V f\|^2 &\leq |(Vf)(0)|^2 + \|C_\varphi V^2 f\|^2 \\ \|C_\varphi V^2 f\|^2 &\leq |(V^2 f)(0)|^2 + \|C_\varphi V^3 f\|^2 \\ &\vdots \\ \|C_\varphi V^n f\|^2 &\leq |(V^n f)(0)|^2 + \|C_\varphi V^{n+1} f\|^2. \end{aligned}$$

Adding these together gives

$$\|C_\varphi f\|^2 \leq \sum_{k=0}^{n} |(V^k f)(0)|^2 + \|C_\varphi V^{n+1} f\|^2.$$

As f is a polynomial, allow n be the degree of f, so that $V^{n+1} f = 0$. Then

$$\begin{aligned} \|C_\varphi f\|^2 &\leq \sum_{k=0}^{n} |(V^k f)(0)|^2 \\ &= \sum_{k=0}^{n} |\hat{f}(k)|^2 \\ &= \|f\|^2. \end{aligned}$$

Thus, C_φ is an H^2-norm contraction when restricted to the set of polynomials in z.

Now suppose f is not a polynomial. Let f_n be the n^{th} partial sum of its Taylor series. Then $f_n \to f$, and $\|f_n\| \leq \|f\|$. Let m and n be positive integers. Then $f_n - f_m$ is a polynomial, and thus $\|(f_n - f_m) \circ \varphi\| \leq \|f_n - f_m\|$. Thus $(f_n \circ \varphi)$ is a Cauchy sequence, and therefore converges to some $g \in H^2$. Because f_n converges in H^2, and hence pointwise on the unit disk $\mathbf{U}$, we see that $g = f \circ \varphi$. Thus, C_φ preserves H^2. Now, as $f_n \circ \varphi \to f \circ \varphi$ and $\|f_n \circ \varphi\| \leq \|f_n\|$, $\|f \circ \varphi\| \leq \|f\|$.

Having proved a composition operator with symbol that vanishes at zero preserves H^2, we will now prove that an arbitrary C_φ preserves H^2. To do this, we will express a given C_φ as $C_\psi C_{\alpha_p}$, where $\psi(0) = 0$ and α_p is a special automorphism on the disk, as defined below.

Definition 2.5: Given $p \in \mathbf{U}$, the *special automorphism* from $\mathbf{U} \to \mathbf{U}$, denoted α_p, is defined by $\alpha_p(z) = \frac{p-z}{1-\bar{p}z}$.

Note the special automorphism interchanges p with 0 (that is $\alpha_p(0) = p$ and $\alpha_p(p) = 0$), and $\alpha_p = \alpha_p^{-1}$.

Lemma 2.6: For all $p \in \mathbf{U}$, C_{α_p} maps H^2 to H^2, moreover C_{α_p} is bounded on H^2 and $\|C_{\alpha_p}\| \leq \sqrt{\frac{1+|p|}{1-|p|}}$.

Proof: Suppose f is holomorphic in a closed neighborhood of the closed unit disk, $R\mathbf{U}$ for $R > 1$. Then $\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$ by theorem 1.14. Using a change of variable, we see

$$\begin{aligned} \|f \circ \alpha_p\|^2 &= \frac{1}{2\pi}\int_0^{2\pi} |f(\alpha_p(e^{i\theta}))|^2 d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 |\alpha_p{}'(e^{it})| dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 \frac{1-|p|^2}{|1-\bar{p}e^{it}|^2} dt \\ &\leq \frac{1-|p|^2}{(|1-|p|)^2} \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 dt \\ &= \frac{1+|p|}{1-|p|}\|f\|^2. \end{aligned}$$

Thus $\|C_{\alpha_p}\| \leq \sqrt{\frac{1+|p|}{1-|p|}}$

Theorem 2.7: If $\varphi : \mathbf{U} \to \mathbf{U}$ is analytic, then $C_\varphi : H^2 \to H^2$, and

$$\|C_\varphi\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

Proof: Note that for $p = \varphi(0)$ and $\psi = \alpha_p \circ \varphi$, ψ goes from $\mathbf{U}$ to $\mathbf{U}$ and fixes 0. By the self-inverse property of α_p, $\varphi = \alpha_p \circ \psi$, or $C_\varphi = C_\psi C_{\alpha_p}$. We have already shown that C_ψ is a bounded contraction, and the product of two bounded operators is bounded by proposition 1.25. Using the just proven lemma, we see $\|C_\varphi\| \leq \|C_\psi\|\|C_{\alpha_p}\| \leq \sqrt{\frac{1+\varphi(0)}{1-\varphi(0)}}$.

Thus C_φ is a bounded linear operator on H^2. Hence, there exists another bounded linear operator C_φ^* such that $\langle C_\varphi f, g\rangle = \langle f, C_\varphi g\rangle$ for all $f, g \in H^2$. In general, determining the image of a H^2 function under C_φ^* is difficult. However, it is less difficult for a reproducing kernel in H^2.

Propostion 2.8: For C_φ and k_α as above, $C_\varphi^* k_\alpha = k_{\varphi(\alpha)}$.

Proof: $\langle f, C_\varphi^* k_\alpha \rangle = (C_\varphi f)(\alpha) = f(\varphi(\alpha)) = \langle f, k_{\varphi(\alpha)} \rangle$.

Thus, C_φ maps the set of Hardy space reproducing kernels into itself. Adjoint compostion operators are the only bounded linear operators on H^2 that have this property.

Theorem 2.9: Suppose $T : H^2 \to H^2$ is a bounded linear operator, and given any $\alpha \in \mathbf{U}$, there exists $\beta \in \mathbf{U}$ such that $Tk_\alpha = k_\beta$. Then $T^* = C_\varphi$ for some $\varphi \in \mathcal{H}(\mathbf{U})$ taking $\mathbf{U}$ into itself.

Proof: Define $\varphi : \mathbf{U} \to \mathbf{C}$ by $\varphi(\alpha) = \langle T^* z, k_\alpha \rangle$. Note $\varphi \in \mathcal{H}(\mathbf{U})$ $(\varphi(\alpha) = (T^* z)(\alpha))$. Let $\alpha_0 \in \mathbf{U}$ be arbitrary. We have

$$\begin{aligned} \varphi(\alpha_0) &= \langle T^* z, k_{\alpha_0} \rangle \\ &= \langle z, T k_{\alpha_0} \rangle \\ &= \langle z, k_{\beta_0} \rangle \\ &= \beta_0, \end{aligned}$$

for some $\beta_0 \in \mathbf{U}$, by hypothesis. Thus, $\varphi(\alpha_0) = \beta_0 \in \mathbf{U}$, and we see that φ is an analytic self-map of $\mathbf{U}$.

Now we must show $C_\varphi = T^*$. Let $f \in H^2$ be arbitrary, and suppose $\alpha \in \mathbf{U}$. We have seen that $Tk_\alpha = k_{\varphi(\alpha)}$. Now, $\langle (C_\varphi - T^*) f, k_\alpha \rangle = \langle f, (C_\varphi^* - T) k_\alpha \rangle = \langle f, k_{\varphi(\alpha)} - Tk_\alpha \rangle = 0$, and thus $(C_\varphi - T^* f)(\alpha) = 0$, and as α is arbitrary, $(C_\varphi - T^* f) = 0$

We have already proven that $\|T^*\| = \|T\|$ for all bounded linear operators, and thus we can readily assert $\|C_\varphi^*\| = \|C_\varphi\|$. We will use this fact in the following section to establish a lower bound on the norms of compostion operators.

We will conclude this section with a brief discussion of the spectrum of compostition operators. Much research has been done in this area. For example, the following is known. Suppose $\varphi \in \mathcal{H}(\mathbf{U})$ is such that $\varphi(\mathbf{U})$ is contained in some polygon contained in $\mathbf{U}$. Then there is a point $\alpha \in \mathbf{U}$ such that $\varphi(\alpha) = \alpha$ and $\sigma(C_\varphi) = \{0\} \cup \{\varphi'(\alpha)^n : n = 0, 1, 2, ...\}$. However, the proof of this result is beyond the scope of this paper. It is easy to establish a special case of this, as we will show below.

Propostion 2.10: For $\varphi(z) = \lambda z$, where $|\lambda| < 1$, $\sigma(C_\varphi) = \{\lambda^n : n = 0, 1, 2, ...\}$.

Proof: For φ as above and $f = z^n$, $C_\varphi f = (\lambda)^n f$. Thus, λ^n is an eigenvalue of C_φ, with eigenvector f. This works for all $n \geq 0$, and thus 0 is also in the spectrum , as the spectrum must be closed. To show that this is the entire spectrum, let us look at $C_\varphi - \alpha I$ for $\alpha \notin \{\lambda^n : n = 0, 1, 2, ...\}$. $C_\varphi - \alpha I$ is invertible, with inverse $T : H^2 \to H^2$ defined as $Tf(z) = \sum_{n=0}^{\infty} \frac{\hat{f}(n) z^n}{\lambda^n - \alpha}$

Section Two: Norms of Composition Operators

In the last section we analyzed some general properties of composition operators. In this section we shall look at some more specific properties of their norms and adjoints.

Theorem 2.11: For $\varphi : \mathbf{U} \to \mathbf{U}$ analytic, and C_φ thus from H^2 to H^2,

$$\sqrt{\frac{1}{1-|\varphi(0)|^2}} \leq \|C_\varphi\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

Proof: We have already seen the proof of the second inequality (Theorem 2.7). For the first, we will look at how C_φ^* operates on the reproducing kernel k_0. Recall that $\|C_\varphi\| = \|C_\varphi^*\|$, from an earlier proof. Then $\|C_\varphi\| \geq \|C_\varphi^* 1\|$, as $\|C_\varphi^*\|$ is a sup. Now,

$$\begin{aligned} \|C_\varphi^* 1\| &= \|C_\varphi^* k_0\| \\ &= \|k_{\varphi(0)}\| \\ &= \sqrt{\langle k_{\varphi(0)}, k_{\varphi(0)}\rangle} \\ &= \left(\frac{1}{1-|\varphi(0)|^2}\right)^{\frac{1}{2}}. \end{aligned}$$

For general symbols φ, these are the best estimates possible, as we will show below.

Lemma 2.12: For the special automorphism α_p on the disk,

$$1-|\alpha_p(z)|^2 = \frac{(1-|p|^2)(1-|z|^2)}{|1-\overline{p}z|^2}.$$

Proof:

$$\begin{aligned} 1-|\alpha_p(z)|^2 &= 1-|\frac{p-z}{1-\overline{p}z}|^2 \\ &= \frac{1}{|1-\overline{p}z|^2}(|1-\overline{p}z|^2-|p-z|^2) \\ &= \frac{1}{|1-\overline{p}z|^2}((1-\overline{p}z)(1-p\overline{z})-(p-z)(\overline{p}-\overline{z})) \\ &= \frac{1}{|1-\overline{p}z|^2}(1+|p|^2|z|^2-|p|^2+|z|^2) \\ &= \frac{1}{|1-\overline{p}z|^2}(1-|p|^2)(1-|z|^2). \end{aligned}$$

Proposition 2.13: For α_p as defined above, $\|C_{\alpha_p}\| = \sqrt{\frac{1+|p|}{1-|p|}}$.

Proof: We have already seen in the previous section that $\|C_{\alpha_p}\| \leq \sqrt{\frac{1+|p|}{1-|p|}}$. We shall begin by looking at $\frac{\|C_{\alpha_p} k_\beta\|}{\|k_\beta\|}$, for $\beta \in \mathbf{U}$.

$$\begin{aligned}\|C_{\alpha_p}\| &\geq \frac{\|C_{\alpha_p} k_\beta\|}{\|k_\beta\|} \\ &= \frac{\|k_{\alpha_p(\beta)}\|}{\|k_\beta\|} \\ &= \left(\frac{1-|\beta|^2}{1-|\alpha_p(\beta)|^2}\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{1-|\beta|^2}{\frac{(1-|p|^2)(1-|\beta|^2)}{|1-\overline{p}\beta|^2}}} \\ &= \sqrt{\frac{|1-\overline{p}\beta|^2}{1-|p|^2}},\end{aligned}$$

and, as $\beta \to \frac{-p}{|p|}$, this becomes $\sqrt{\frac{1+|p|}{1-|p|}}$.

Proposition 2.14: The lower bound estimate is the best possible.

Proof: Let $\varphi \in H^2$ be defined by $\varphi(z) = p$ for some $p \in \mathbf{U}$. Then for $f \neq 0$ in H^2,

$$\begin{aligned}\frac{\|C_\varphi f\|}{\|f\|} &= \frac{|f(p)|}{\|f\|} \\ &= \frac{|\langle f, k_p\rangle|}{\|f\|} \\ &\leq \frac{\|f\|\|k_p\|}{\|f\|} \\ &= \|k_p\| \\ &= \sqrt{\frac{1}{1-|p|^2}}.\end{aligned}$$

Thus $\|C_\varphi\| \leq \sqrt{\frac{1}{1-|p|^2}} = \sqrt{\frac{1}{1-|\varphi(0)|^2}}$, and, as $\sqrt{\frac{1}{1-|\varphi(0)|^2}}$ is also a lower bound on $\|C_\varphi\|$, it must be the norm.

If we restrict our attention to certain classes of symbols, we can get better estimates, as we shall examine in the next section.

Section Three: Norm Calculations

Recall that in the previous sections we established, by proving $\|C_\varphi\| = \|C_\varphi^*\|$ and $\|C_\varphi^* k_0\| = \frac{1}{\sqrt{1-|\varphi(0)|^2}}$, that $\|C_\varphi\| \geq \sqrt{\frac{1}{1-|\varphi(0)|^2}}$. Generally, we have $\|C_\varphi\| \geq \sup\{\|C_\varphi^* \tilde{k}_\alpha\|\}$, where $\tilde{k}_\alpha$ is the nomalized reproducing kernel at α (that is $\|\tilde{k}_\alpha\| = 1$). Thus,

$$\begin{aligned}\|C_\varphi\| &\geq \sup\left\{\sqrt{\frac{1-|\alpha|^2}{1-|\varphi(\alpha)|^2}} : \alpha \in \mathbf{U}\right\} \\ &= \sup\left\{\sqrt{\frac{1-r^2}{1-|\varphi(re^{i\theta})|^2}} : 0 \leq r < 1, 0 \leq \theta < 2\pi\right\}.\end{aligned}$$

In this section, we will use these results to establish a better lower bound for the norm of a certain class of composition operators.

We shall begin by defining a class of function which we will work with throughout this section.

Definition 2.15: Let $a, b, c, d \in \mathbf{C}$, with $d \neq 0$. A *linear-fractional transformation* (frequently written LFT) $\psi : \mathbf{C} \to \mathbf{C}$ is a function of the form $\psi(z) = \frac{az+b}{cz+d}$.

In general, the calculation of norms of composition operators can be very complicated, and yield results that are not easy to interpret or work with. The following theorem will illustrate this by displaying the norm of a very simple class of compostion operators (see [2, theorem 3]).

Theorem 2.16: Let $\varphi(z) = sz + t$, where $s, t \in \mathbf{C}$, and $|s| + |t| \leq 1$. Then

$$\|C_\varphi\| = \sqrt{\frac{2}{1+|s|^2-|t|^2+\sqrt{(1-|s|^2+|t|^2)^2-4|t|^2}}}.$$

Due to the difficulties in determining the norms of composition operators, we shall attempt the easier task of improving the lower bound for the norm of composition operators with linear-fractional symbol of the form $\frac{b}{cz+d}$. First we will prove the following proposition, in order to facilitate this analysis.

Proposition 2.17: A non-constant linear-fractional transformation of the form $\psi(z) = \frac{b}{cz+d}$ is an analytic self-map of $\mathbf{U}$ if and only if $|b| + |c| \leq |d|$.

Proof: Note that in general, b and c cannot be equal to zero, because if they were, then ψ would be a constant function (with value 0 or $\frac{b}{d}$).

We shall first prove that if $\psi(z) = \frac{b}{cz+d}$ is an analytic self map of $\mathbf{U}$ then $|b|+|c| \leq |d|$. As ψ is analytic, $z \mapsto cz + d$ has no zeros over $\mathbf{U}$, and therefore $|c| \leq |d|$, or $||d| - |c|| =$

$|d| - |c|$. By definition of self-map, $|\psi(z)| = |\frac{b}{cz+d}| < 1$ for all $z \in \mathbf{U}$. Note that the continuity of $z \mapsto |cz+d|$ gives that $|b| \leq |cz+d|$ for all z in the closed unit disk. Thus, for $z = \frac{-|c|d}{|d|c}$ on the unit circle,

$$\begin{aligned} |b| &\leq |cz+d| \\ &= |c\frac{-|c|d}{|d|c} + d| \\ &= |d - \frac{|c|d}{|d|}| \\ &= ||d| - |c|||\frac{d}{|d|}| \\ &= |d| - |c|. \end{aligned}$$

Thus, $|d| \geq |c| + |b|$.

Now assume that $|b| + |c| \leq |d|$. Then $\psi(z) = \frac{b}{cz+d}$ is analytic over $\mathbf{U}$ because $z \mapsto cz+d$ has no zeros for $z \in \mathbf{U}$, as $|c| < |d|$ (recall $b \neq 0$ because ψ is nonconstant). Now, for $z = re^{i\theta}$ where $r < 1$,

$$\begin{aligned} |\frac{b}{cz+d}| &= \frac{|b|}{|cz+d|} \\ &\leq \frac{|b|}{||d| - |c|r|} \\ &< \frac{|b|}{|d| - |c|} \\ &\leq \frac{|d| - |c|}{|d| - |c|} \\ &= 1. \end{aligned}$$

The second (strict) inequality is true because $c \neq 0$. Therefore, $\psi(z)$ is an analytic self-map of $\mathbf{U}$.

We will now examine the bounds on $\|C_\psi\|$ where $\psi(z) = \frac{b}{cz+d}$ is an analytic self-map of $\mathbf{U}$. We will do this by finding the supremum of $\frac{1-r^2}{1-|\varphi(re^{i\theta})|^2}$ over r and θ. We will now proceed by looking at $\frac{1}{1-|\psi(re^{i\theta})|^2}$, and establishing a maximum over θ for fixed r. For $\varphi(z) = \frac{b}{cz+d}$ and $z = re^{i\theta}$,

$$\frac{1}{1 - |\psi(re^{i\theta})|^2} = \frac{1}{1 - |\frac{b}{cre^{i\theta}+d}|^2}.$$

We must therefore minimize $|cre^{i\theta} + d|$ over θ.

$$|cre^{i\theta} + d| \geq ||d| - |c|r| = |d| - r|c|,$$

with the last equality proven in the preceding proposition. This is a minimum because for θ such that $\theta+\arg(c)=\arg(d)+\pi$, $|cre^{i\theta}+d|$ will be minimized, and will equal $|d|-r|c|$. The fraction thus becomes

$$\frac{1}{1-\frac{|b|^2}{(|d|-|c|r)^2}},$$

and the lower bound on the norm is thus

$$\sup\{\sqrt{\frac{(1-r^2)(|d|-|c|r)^2}{(|d|-|c|r)^2-|b|^2}}:0\leq r<1\}. \tag{2.18}$$

Note then that to get a lower bound of the norm of $\|C_\varphi\|$, we must compute the supremum over r of the fraction within the radical in the expression above. Using the basic methods of calculus, this is equivalent to finding the zeros of the fraction below, which is the derivative of the fraction in the radical above,

$$\frac{2(|d|-|c|r)(|c||b|^2+|d||b|^2r-|d|^3r-2|c||b|^2r^2+3|c||d|^2r^2-3|d||c|^2r^3+|c|^3r^4)}{(|b|^2-|d|^2+2|c||d|r-|c|^2r^2)^2.}$$

This is the same as finding the zeros of $q(r)=|c||b|^2+|d||b|^2r-|d|^3r-2|c||b|^2r^2+3|c||d|^2r^2-3|d||c|^2r^3+|c|^3r^4$ for $0\leq r<1$. This can be done using Ferrari's formula, however the result is too long and complicated to be of any use. We will therefore look at supremums for specific values of $b,c,$ and d.

Examples 2.19: Let $\varphi(re^{i\theta})=\frac{b}{cre^{i\theta}+d}$.

1) Using formula 2.18, we see that when $b=2$, $c=-1$, and $d=3$,

$$\begin{aligned}\|C_\varphi\| &\geq \sqrt{\frac{(1-r^2)(3-r)^2}{(3-r)^2-4}}\\ &= \sqrt{\frac{-r^4+6r^3-8r^2-6r+9}{r^2-6r+5}}.\end{aligned}$$

To find the bound, we must determine the maxima of this last term, which, using the basic methods of calculus, becomes a problem of finding the zeros of $12-25r+10r^2-r^3$. These are 3, $\frac{7+\sqrt{33}}{2}$, and $\frac{7-\sqrt{33}}{2}$. Clearly $1<3, \frac{7+\sqrt{33}}{2}$. Thus, the maximum we seek is at $r=\frac{7-\sqrt{33}}{2}$, where the value of the bound is $\sqrt{\frac{447-79\sqrt{33}}{\sqrt{33}-9}}$, or approximately $\sqrt{2.095}$. Note we know this critical point yields the supremum because at $r=0$, the derivative of $\frac{(1-r^2)(3-r)^2}{(3-r)^2-4}$ is positive ($\frac{24}{25}$), whereas at $r=1$, the derivative is negative ($-\frac{1}{2}$). Note that this value of $\sqrt{2.095}$ is larger than the estimate given by $\|C_\varphi\|\geq\frac{1}{\sqrt{1-|\varphi(0)|^2}}$, which yields $\sqrt{\frac{9}{5}}=\sqrt{1.800}$.

2) Again, using formula 2.18, we see that for $b = 2$, $c = 4 + 3i$, and $d = 7$,

$$\|C_\varphi\| \geq \sqrt{\frac{(1-r^2)(7-5r)^2}{(7-5r)^2-4}}.$$

Finding a maximum of this over r using calculus requires finding the zeros of $28 - 405r + 450r^2 - 125r^3$, which are $r = \frac{7}{5}$, $r = \frac{11+\sqrt{105}}{10}$, and $r = \frac{11-\sqrt{105}}{10}$. Once again, only $0 \leq \frac{11-\sqrt{105}}{10} < 1$, and thus the maximum is at approximately $r = .0753$, where the value of the bound is approximately $\sqrt{1.094}$. Note that the value of the derivative of $\frac{(1-r^2)(7-5r)^2}{(7-5r)^2-4}$ at zero is $\frac{56}{405}$, and the value of the derivative at 1 is $-\frac{13}{10}$, and thus we know the critical point at $r = .0753$ is the supremum. The estimate for the bound of this operator using the method of the last section is $\sqrt{1.088}$. Thus we see that this bound is not always a great improvement over the original bound estimate, but in some cases is a significant improvement.

We will now present a proof that the above estimate is always better than the bound established in the last section.

Theorem: Let b, c, d be non-zero elements in $\mathbf{U}$ be such that $|d| \geq |b| + |c|$. For $\psi : \mathbf{U} \to \mathbf{U}$ defined by $\psi(z) = \frac{b}{cz+d}$,

$$\sqrt{\frac{1}{1-|\psi(0)|^2}} < \sup\left\{\sqrt{\frac{(1-r^2)(|d|-|c|r)^2}{(|d|-|c|r)^2-|b|^2}} : 0 \leq r < 1\right\}.$$

Proof: For any ψ of this form, $|\psi(0)|^2 = |\frac{b}{d}|^2$, and thus,

$$\sqrt{\frac{1}{1-|\psi(0)|^2}} = \sqrt{\frac{|d|^2}{|d|^2-|b|^2}}.$$

We will now analyze $\sup\{\frac{(1-r^2)(|d|-|c|r)^2}{(|d|-|c|r)^2-|b|^2} : 0 \leq r < 1\}$. We will proceed by looking at the values of $\frac{(1-r^2)(|d|-|c|r)^2}{(|d|-|c|r)^2-|b|^2}$ near $r = 0$. At $r = 0$, $\frac{(1-r^2)(|d|-|c|r)^2}{(|d|-|c|r)^2-|b|^2} = \frac{|d|^2}{|d|^2-|b|^2}$, which is equal to the estimate from the previous chapter. The derivative at $r = 0$ is $\frac{2|b|^2|c||d|}{(|d|^2-|b|^2)^2}$, which is positive, and thus the function is increasing in some neighborhood of $r = 0$, from the continuity of the derivative. Thus, there exists $\epsilon > 0$ such that for $r = \epsilon$, $\frac{(1-r^2)(|d|-|c|r)^2}{(|d|-|c|r)^2-|b|^2} > \frac{|d|^2}{|d|^2-|b|^2}$, and hence the theorem is proven, as there exists an element of $\{\frac{(1-r^2)(|d|-|c|r)^2}{(|d|-|c|r)^2-|b|^2} : 0 \leq r < 1\}$ which is greater than $\frac{|d|^2}{|d|^2-|b|^2}$, and therefore the supremum is greater as well.

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