<u>Rado's Selection Principle:</u>
<u>Equivalences and Applications</u>

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Chapter 1: Introduction

Rado's Selection Principle is a combinatorial theorem which allows the characterization of infinite objects (e.g. graphs, groups, partially-ordered sets) based on the characterization of their finite subparts. That is, a typical result of the application of Rado's Selection Principle would be a theorem of the following sort: Object A has property P if and only if every finite subobject of A has property P. The necessity of the second hypothesis is usually obvious because the subobjects usually inherit the properties of the objects (in workable applications), so Rado is used to prove sufficiency. Theorems of this sort are extremely useful because it is normally possible to check directly a condition on a finite object, and impossible to do so on an infinite one. This description is, of course, far too general, but it gives some indication of types of problems here undertaken.

This study of Rado's Selection Principle originates from the so-called Marriage Problem which is the name sometimes given to Philip Hall's Theorem on systems of distinct representatives. Basically, Hall's Theorem is a result concerning choice functions, and it is appropriate now to give a few definitions.

Definition. Let $\hat{A}=\{A_i : i \in I\}$ be an indexed family of sets where $|I|$ is any cardinal. Then $f:\hat{A} \to \cup\hat{A}$ is a choice function if $\forall i \in I \quad f(A_i) \in A_i$.

Definition. A system of representatives (SR) for a family of sets is the range of a choice function on that family, i.e. given f, a choice function, $SR=\{f(A_i) : i \in I\}$. The system is said to be distinct (SDR) if $\forall i,j \in I$, $i \neq j$, $f(A_i) \neq f(A_j)$.

Definition. An indexed family of sets, $\hat{A}=\{A_i : i \in I\}$, satisfies the Hall condition if and only if $\forall J \subseteq I$, J:finite, $|A_J| \geq |J|$ where $A_J=\{a : \exists i \in J \text{ s.t. } a \in A_i\}$.

Hall's Theorem, then , says:

Theorem. (P. Hall) Let $\hat{A}=\{A_i : 1 \leq i \leq n\}$ be a finite family of finite sets. Then $\hat{A}$ has an SDR if and only if $\hat{A}$ satisfies the Hall condition.

The necessity of the condition follows immediately from the pigeonhole principle: If there were a finite subfamily J with $|J|=m$ and $|A_J|<m$, $f|A_J$ would be a surjection but not an injection, and the representatives would not be distinct.

Sufficiency is more difficult, but accessible proofs are given by Halmos and Vaughn and by Rado.[1]

If the Marriage Problem (or Theorem) is extended to an infinite family of finite sets, it turns out that the same result holds, i.e. $\hat{A}$ has an SDR if and only if it satisfies the Hall condition. Stated more colorfully, the Marriage Problem says that if each of a (possibly infinite) set of boys is acquainted with a finite set of girls, each boy can marry one of his acquaintances if and only if each set of n boys collectively knows at least n girls, for all possible n, finite. Mathematically, each boy is a member of the index set I, and A_i is the set of boy_i's acquaintances.

The infinite extension of Hall's Theorem does not necessarily hold if any of the sets in the infinite family is itself infinite. This is easily demonstrated by the following counterexample:

Example. Let $G_0=\{1,2,3,...\}$ and let $G_i=\{i\}$. Although $\hat{G}=\{G_0\}\cup\{G_i\}$ satisfies the Hall condition, it clearly does not have an SDR because if $f:\hat{G}\to\cup\hat{G}$ is any choice function, then the restriction $f:\hat{G}\setminus\{G_0\}\to\cup\hat{G}$ is surjective and thus $\exists i$ such that $f(G_0)=f(G_i)$.

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[1] Philip F. Reichmeider, The Equivalence of Some Combinatorial Matching Theorems, (Washington, New Jersey: Polygonal Publishing House, 1984), p.38-9.

The Marriage Theorem is proved by Halmos and Vaughn using Tychonoff's Theorem from topology,[2] but it is also a very natural example of an application of Rado's Selection Principle which is weaker than Tychonoff's Theorem (see below). Thus, we state Rado's Selection Principle:

<u>Theorem</u>. (R. Rado) Let $\hat{G}=\{G_i : i\epsilon I\}$ be an indexed family of finite subsets of a set W, and let $\hat{J}$ denote the class of all finite subsets of the index set I. For each $J\epsilon\hat{J}$, let f_J be a choice function on the subfamily $\{G_i : i\epsilon J\}$. Then there exists a choice function f on the entire family $\hat{G}$ such that, for each $J\epsilon\hat{J}$, there exists a set $K\epsilon\hat{J}$ with $J\subseteq K$ and $f(G_i)=f_K(G_i)$ for all $i\epsilon J$.

By way of explanation, $\hat{G}$ is an indexed family of sets as in Hall's Theorem, and W is any set containing $\cup\hat{G}$. For every finite subfamily J, a <u>local</u> choice function f_J is defined. Then the conclusion is that there exists a <u>global</u> choice function f which, for every finite subfamily, coincides on that subfamily (as a subdomain) with one of the local choice functions whose domain contains that subfamily.

As indicated, Rado's Selection Principle can be used to prove The Marriage Theorem (infinite version):

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[2] Halmos and Vaughn, "The Marriage Problem", <u>The American Journal of Mathematics</u>, v.72 (1950), pp.214-215

Proof: Let I be the set of boys, and for each boy i let B_i be the set of girls he is acquainted with. Since $\mathcal{B}=\{B_i : i\in I\}$ satisfies the Hall Condition by hypothesis, for each finite C in $\mathcal{B}$ we may apply Hall's Theorem (for finite families) and define a local choice function f_C such that $f_C(B_i)\neq f_C(B_j)$ for $i,j\in C$, $i\neq j$. Then $\exists f:\mathcal{B}\rightarrow \cup\mathcal{B}$, a global choice function satisfying Rado. We must show range(f) to be an SDR; that is, we must show that f is an injection. Consider $D=\{B_i, B_j\}$ where $i\neq j$. Then $\exists C\subseteq\mathcal{B}$, $C\supseteq D$, C:finite such that $f(B_i)=f_C(B_i)$ and $f(B_j)=f_C(B_j)$. But since $f_C(B_i)\neq f_C(B_j)$, we get $f(B_i)\neq f(B_j)$ and f is an injection. QED

Chapter 2: Equivalences

Rado's Selection Principle is true in the presence of the Axiom of Choice for Finite Sets, and its proof by Gottschalk uses Tychonoff's Theorem from topology in much the same way that Halmos and Vaughn use it to prove the Marriage Theorem. However, what is really needed is not the full power of Tychonoff's Theorem but only that of a weaker version of the theorem - Tychonoff's theorem for products of finite spaces, hereafter referred to as Tychonoff Finite, which says:

Theorem. (Tychonoff Finite) If $\{X_i : i\in I\}$ is a family of nonempty finite topological spaces, then $X=\Pi_{i\in I}X_i$ is compact in the product topology.

Definition. A topological space X is compact if and only if for every collection of open sets $\hat{O}=\{O_k \mid O_k\subseteq X,\ O_k\text{:open},\ k\in K\}$ which covers X (i.e. $X\subseteq \cup\hat{O}$), there exists a finite subcollection $\hat{O}'\subseteq\hat{O}$ such that $\hat{O}'$ covers X.

Definition. Given a family $\{X_i : i\in I\}$ of topological spaces, a set $Y=\Pi_{i\in I}Y_i$ is called open in the product topology on the product space $X=\Pi_{i\in I}X_i$ if and only if there exists a finite set $J\subseteq I$ such that $Y_i\subseteq X_i$ is open $\forall i\in J$ and $Y_i=X_i$ $\forall i\in(I\setminus J)$.

Before we prove Rado using Tychonoff Finite, we need two more definitions and the statement of the Axiom of Choice for Finite Sets:

Definition. A family $\{X_i : i \in I\}$ of nonempty closed subsets of a topological space X possesses the finite intersection property (f.i.p.) if and only if every finite subfamily has nonempty intersection. That is, $\forall J \subseteq I$, J:finite, $\cap_{i \in J} X_i \neq \phi$.

Definition. A topological space X is said to have the discrete topology if and only if every subset is both open and closed.

Axiom of Choice for Finite Sets. (ACF) There exists a choice function for every family of non-empty finite sets.

We now have the equipment with which to prove Rado's Selection Principle, which we restate for convenience:

Theorem. (R. Rado) Let $\hat{G} = \{G_i : i \in I\}$ be an indexed family of finite subsets of a set W, and let $\hat{J}$ denote the class of all finite subsets of the index set I. For each $J \in \hat{J}$, let f_J be a choice function on the subfamily $\{G_i : i \in J\}$. Then there exists a choice function f on the entire family G' such that, for each $J \in \hat{J}$, there exists a set $K \in \hat{J}$ with $J \subseteq K$ and $f(G_i) = f_K(G_i)$ for all $i \in J$.

Proof: Endow each G_i with the discrete topology; thus, since each G_i is finite and every subset is open, the product space $G=\Pi_{i\epsilon I}G_i$ is compact, by Tychonoff Finite. $\forall J\subseteq\hat{J}$ let $E_J=\{f|f\epsilon G, \exists K\text{:finite}, J\subseteq K \text{ such that } f(G_i)=f_K(G_i)\ \forall i\epsilon J\}$. Each E_J is non-empty by ACF.

To show E_J closed: Let $f\epsilon G\backslash E_J=\{f|\forall K\supseteq J\ \exists j\epsilon J \text{ such that } f(G_j)\neq f_K(G_j)\}$. Let $H=\{h|h(G_i)=f(G_i)\ \forall i\epsilon J\}$. Then $f\epsilon H$ because $f(G_i)=f(G_i)\ \forall i\epsilon J$. $H\subseteq G\backslash E_J$ because $\forall K\ \exists i$ such that $h(G_i)=f(G_i)\neq f_K(G_i)$. $H=\cap_{i\epsilon J}P_i^{-1}(f(G_i))$ (where P_i is the projection onto the i-th coordinate space), a finite intersection of open sets, so H is open. Thus, $\forall f\epsilon G\backslash E_J$, $f\epsilon H$:open, and thus $G\backslash E_J$:an open neighborhood and E_J:closed.

Then $\hat{E}=\{E_J|J\epsilon\hat{J}\}$ is a family of nonempty closed sets. To show that $\hat{E}$ has the finite intersection property: Let $\hat{E}'$ be a finite subcollection of $\hat{E}$, and let $\hat{J}'$ be the corresponding finite subcollection of $\hat{J}$. Then $\cap\hat{E}'=\{f|\exists K\supseteq\cup\hat{J}' \text{ such that } f(G_i)=f_K(G_i)\ \forall i\epsilon\cup\hat{J}'\} = E_{\cup\hat{J}'}\ \epsilon\ \hat{E}$, and thus $\cap\hat{E}'$ is nonempty. Thus $\hat{E}$ has the finite intersection property.

Since G is compact and $\hat{E}$ has f.i.p., $\cap\hat{E}\neq\phi$ (see Theorem 1 below) and any $f\epsilon\cap\hat{E}$ is a global choice function satisfying the conclusions of Rado's Selection Principle. QED[3]

Thus have we seen that in the presence of ACF, Rado's Selection Principle is a consequence of Tychonoff Finite, but in

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[3] W.H. Gottschalk, "Choice Functions and Tychonoff's Theorem," Proceedings of the AMS, v.2 (1951), p.172

fact, in the presence of ACF, these two theorems are equivalent not only to each other, but also to the Alexander Subbase Theorem from topology. Alexander's Theorem can be proved using the full Axiom of Choice, but of course our present interest is in proving it using only Rado. Before doing so and then showing that Alexander's Theorem implies Tychonoff Finite, it is necessary to have a more complete explanation of the topology involved.

Definition. A collection of open sets $\hat{S}$ in a topology T is called a subbase for the topology if and only if every open set in T is the arbitrary union of finite intersections of sets in $\hat{S}$.

Using deMorgan's Laws, we may similarly define a collection of closed sets $\hat{C}$ to be a subbase for the closed sets of the topology if and only if every closed set in T is the arbitrary intersection of finite unions of sets in $\hat{C}$.

The concept of a subbase allows us to give an alternative definition of the product topology which may prove useful when later discussing Tychonoff's Theorem:

Definition. Given a family $\{X_a : a \in A\}$ of topological spaces and the product space $X = \Pi_{a \in A} X_a$, let a subbase for the product topology be the collection of sets $\hat{S} = \{Y \mid \exists a \text{ s.t. } Y = P_a^{-1}(U)$ for some $U \subseteq X_a$, where $P_a : X \rightarrow X_a$ is a projection$\}$. Then an open set in the product topology is an arbitrary union of finite intersections of elements of the subbase.

Theorem 1. A topological space is compact if and only if every family of closed subsets with the finite intersection property has nonempty intersection. (Thus, we may use this as an alternate definition of compactness.)[4]

Note that this last theorem has already been cited in the proof of Rado above. We are now ready to prove Alexander's Theorem which refines the sufficiency condition of the previous theorem to a condition only on a subbase for the closed sets instead of on all families of closed sets.

Theorem. (Alexander) A topological space X is compact if there exists a subbase $\hat{S}$ for the closed sets of X such that every subfamily of $\hat{S}$ with the finite intersection property has nonempty intersection.

Proof: Let X be a topological space, and suppose $\hat{S}$ is a subbase for the closed sets with the stated property. Using Theorem 1 to show X is compact, we let $\hat{G}$ be a family of closed subsets of X with f.i.p. and show that $\cap\hat{G}\neq\phi$.

Define $\Gamma=\{F \mid F\text{:finite subfamily of } \hat{S},\ \exists G\epsilon\hat{G} \text{ s.t. } G\subseteq \cup F\}$.

$\beta=\{B \mid \exists F\epsilon\Gamma \text{ s.t. } B=\cup F\}$

Suppose $x\notin\cap\hat{G}$. Then $\exists A\epsilon\hat{G}$ with $x\notin A$. Since $\hat{S}$ is a subbase,

[4] John L. Kelley, General Topology, (New York: Springer-Verlag, 1955), p.136

$\exists F\subseteq\hat{S}$, F:finite s.t. $A\subseteq \cup F$ and $x\notin \cup F$ (otherwise since $A=\cap\{\cup F\}$, if there is no F s.t. $x\notin\cup F$, then $x\in A$). Thus, $x\notin\cap B$, and thus, $\cap\beta\subseteq\cap\hat{G}$. It will be sufficient to show that $\cap\beta\neq\phi$.

We aim at applying Rado to the family Γ.

Let: $P=\{F_1,F_2,\ldots F_k\}\subseteq\Gamma$

$F_i=\{S_{i1},S_{i2},\ldots S_{in_i}\}$

$B_i=\cup F_i=S_{i1}\cup S_{i2}\cup\ldots\cup S_{in_i}$

Since β has f.i.p. ($\beta\supseteq\beta'$:finite $\Rightarrow$ $\cap\beta'\supseteq\cap\{\cup F\}\supseteq\cap\hat{G}\neq\phi$ since $\hat{G}$ has f.i.p.), $\cap\{B_i \mid 1\leq i\leq k\}\neq\phi$ (*).

Let $H=\{(h_1,h_2,\ldots h_k) \mid 1\leq h_i\leq n_i$ for all $i=1,\ldots k\}$

Then $\cap_i B_i=\cap_i[\cup_j S_{ij}]=\cup_{h\in H}[S_{1h_1}\cap S_{2h_2}\cap\ldots\cap S_{kh_k}]$

Since $\cap\{B_i\}\neq\phi$, $\exists h\in H$ s.t. $S_{1h_1}\cap S_{2h_2}\cap\ldots\cap S_{kh_k}\neq\phi$.

Then $\forall P\subseteq\Gamma$, P:finite define $f_P:P\rightarrow\hat{S}$ by $f_P(F_i)=S_{ih_i}\in F_i$ where $h=(h_1,h_2,\ldots h_k)$ is chosen by the Axiom of Choice for finite sets such that $\cap\{S_{ih_i} \mid i=1,\ldots k\}\neq\phi$.

Then by Rado, $\exists f:\Gamma\rightarrow\hat{S}$ such that for $\Gamma\supseteq P$:finite, $\exists Q\supseteq P$ and Q:finite such that $f(F)=f_Q(F)$ $\forall F\in P$. Then given P:finite, $\exists Q$:finite with $P\subseteq Q\subseteq\Gamma$ and $\cap\{f(F) \mid F\in P\}=\cap\{f_Q(F) \mid F\in P\}\neq\phi$ since $\{f_Q(F) \mid F\in\Gamma\}$ has f.i.p. Thus, $\{f(F) \mid F\in\Gamma\}$ has f.i.p. since P was arbitrary. Since $f(F)\in\hat{S}$ $\forall F\in\Gamma$, by hypothesis, $\cap\{f(F) \mid F\in\Gamma\}\neq\phi$, and since $\cap\{f(F) \mid F\in\Gamma\}\subseteq\cap\beta$, we get $\cap\beta\neq\phi$.

Thus, $\cap\hat{G}\neq\phi$ and X is compact. QED[5]

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[5] E.S. Wolk, "On Theorems of Tychonoff, Alexander, and R. Rado," Proceedings of the AMS, v.18, Number 1 (1967), p.113-115

Finally, to complete the proof of the equivalence of these three theorems, we must show that Alexander's Subbase Theorem implies Tychonoff Finite. This proof will require a different version of Alexander's Subbase Theorem which is stated in terms of open sets rather than closed sets:

<u>Theorem</u>. (Alexander) A topological space X is compact if there exists a subbase $\hat{S}$ for the open sets of X such that every cover of X by a subfamily of $\hat{S}$ has a finite subcover.

The equivalence of the two versions follows quickly from taking the elements of the subbase for the open sets to be the complements of the elements of the subbase for the closed sets. Thus, a family of closed sets having a nonempty intersection is equivalent to the corresponding family of open sets not covering the space.

We are now ready to prove Tychonoff Finite which is restated for convenience:

<u>Theorem</u>. (Tychonoff Finite) The product of a family of nonempty finite topological spaces is compact in the product topology.

<u>Proof</u>: Let $X=\Pi_{a\epsilon A}X_a$ where each X_a is a finite topological space and X has the product topology.

Let $\hat{S}=\{P_a^{-1}[U] \mid P_a:X\rightarrow X_a$ is a projection and $U\subseteq X_a$ is open$\}$.

To show X to be compact, we use Alexander by choosing a subfamily $\hat{A}\subseteq\hat{S}$ and showing that if all finite subfamilies of $\hat{A}$

fail to cover X, then Â fails to cover X.

$\forall a \in A$ let $B_a = \{U \mid U \subseteq X_a,\ P_a^{-1}[U] \in \hat{A}\}$

Then, by hypothesis, no finite subfamily of B_a covers X_a. By the compactness (finiteness) of X_a $\forall a \in A$, $\exists x_a \in X_a$ such that $\forall U \in B_a$, $x_a \in X_a \setminus U$. If there are more than one such x_a, then we choose one by the Axiom of Choice for Finite Sets. Thus, the point x whose a-th coordinate is x_a does not belong to any member of Â (i.e. $x \notin U\hat{A}$), and Â does not cover X. QED[6]

Rado's Selection Principle and the two equivalent theorems are weaker corollaries to the full version of Tychonoff's Theorem and the full Axiom of Choice:

Theorem. (Tychonoff) The product of a family of nonempty compact topological spaces is compact in the product topology.

The difference between the finite and full versions of the theorem is the condition on the spaces that constitute the product; in the finite version they are finite, and in the full version they are compact. Tychonoff Finite is an immediate consequence of Tychonoff's Theorem because any finite space is necessarily compact.

The Axiom of Choice has many corollaries, both direct (ACF) and indirect (Rado, Alexander, etc.), as well as having many

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[6] Kelley, p.143 (Theorem 13)

different statements, the equivalence of which are beyond the range of this paper.[7] The particular statement here used is so selected because it is most directly shown to follow from Tychonoff's Theorem. Thus, we give the statement and its proof:

<u>Axiom of Choice</u>. If $\{X_a : a \in A\}$ is a family of nonempty sets, then the Cartesian product $\Pi_{a\in A} X_a$ is nonempty.

<u>Proof</u>: $\forall a \in A$, adjoin a single point to each X_a. Let $Y_a = X_a \cup \{*\}$. Define a topology on Y_a by $T_a = \{Y_a, \phi, \{*\}, Y_a \backslash F \mid \forall F \subseteq Y_a, F\text{:finite}\}$. That is, topologize each set with the finite complement topology modified by the addition of the singleton $\{*\}$ as an open set. This is a topology: $Y_a \cap T = T$, $\phi \cap T = \phi$, $\{*\} \cap Y_a \backslash F = \phi$ or $\{*\}$, and $(Y_a \backslash F_1) \cap (Y_a \backslash F_2) = Y_a \backslash (F_1 \cup F_2)$ so T_a is closed under finite intersections, and $\cup_F Y_a \backslash F = Y_a \backslash \cap_F F \in T_a$, so T_a is closed under arbitrary unions. A space with the finite complement topology is known to be compact, and each Y_a is similarly (we need add just one open set containing $*$ to the cover, and if was finite previously, it will remain so).

Let $Y = \Pi_{a\in A} Y_a$. $\forall a \in A$, let $P_a : Y \rightarrow Y_a$ be the usual projective map. Let $Z_a = \{U \mid U \in Y,\ P_a(U) \in X_a\}$; that is, let $Z_a = P_a^{-1}(X_a)$. Since $\{*\}$ is open in Y_a, $Y_a \backslash \{*\} = X_a$ is closed in Y_a, and since projections are continuous, $P_a^{-1}(X_a) = Z_a$ is closed in Y.

Next, we demonstrate that $Z = \{Z_a : a \in A\}$ has f.i.p.: Consider $B \subseteq A$, B:finite. Then choose $x = \{x_a : a \in A\}$ by the

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[7] Herbert B. Enderton, <u>Elements of Set Theory</u>, (Orlando, Florida: Academic Press, 1977), pp.151-153

following: $\forall a \epsilon B$, let $x_a \epsilon X_a$ (this is a finite choice and does not require the use of the Axiom of Choice) and $\forall a \epsilon A \backslash B$, let $x_a = *$ (no use of AC). Then $x \epsilon \cap_{a\epsilon B} Z_a$ and so Z has f.i.p.

Then since Y is compact (by Tychonoff), $\cap Z \neq \phi$ (by Theorem 1). But $\cap Z = \Pi_{a\epsilon A} X_a$, and so $\Pi_{a\epsilon A} X_a \neq \phi$. QED[8]

To show that the Axiom of Choice implies Tychonoff's Theorem, we will use an indirect proof that works from the former to the latter via the Alexander Subbase Theorem. We will prove AC => Alexander using a different 'version' of the Axiom of Choice, namely Zorn's Lemma. The equivalence of these two statements is well known,[9] and we provide the following definitions to prepare for the statement of the lemma:

Definition. A collection of sets $\hat{S}$ is called a chain if and only if $\forall U, V \epsilon \hat{S}$, either $U \subseteq V$ or $V \subseteq U$. That is, set inclusion is a total ordering in a chain.

Definition. A set M is called maximal in a collection of sets $\hat{S}$ if and only if $\forall U \epsilon \hat{S}$, $M \not\subseteq U$.

We may now state the lemma:

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[8] John L. Kelley, "The Tychonoff Product Theorem Implies the Axiom of Choice," Fundamenta Mathematicae, v.37 (1950)

[9] Enderton, pp.151-153,196-199

Zorn's Lemma: Let $\hat{A}$ be a family of sets partially ordered by set theoretic inclusion such that for every chain $\beta \subseteq \hat{A}$, $\cup\beta \in \hat{A}$. Then $\hat{A}$ contains a maximal element M.

After one more topology definition, we demonstrate that the Axiom of Choice implies the Alexander Subbase Theorem:

Definition. A collection of open sets β in a topology T is called a base for the topology if and only if every open set in T is the arbitrary union of sets in β. Thus, a base for the open sets may be generated by taking all finite intersections of sets in a subbase for the open sets.

Alternatively, using deMorgan's Laws, we may define a collection of closed sets β to be a base for the closed sets of the topology if and only if every closed set in T is the arbitrary intersection of sets in β. Thus, a base for the closed sets may be generated by taking all finite unions of sets in a subbase for the closed sets.

Theorem. (Alexander) A topological space X is compact if there exists a subbase $\hat{S}$ for the closed sets of X such that every subfamily of $\hat{S}$ with the finite intersection property has nonempty intersection.

Proof: Assume that $\hat{S}$ has the stated property.

Let B denote the base generated by $\hat{S}$.

If we show that $\forall Q \subseteq B$, if $G \subseteq Q$ and $\cap G \neq \phi$ imply $\cap Q \neq \phi$, then by Theorem 1, we have our result.

Let $Q \subseteq B$ have f.i.p. (a restatement of the 'if' in the previous line)

Let $P=\{E | Q \subseteq E \subseteq B,$ E has f.i.p.$\}$

For any chain C in P, let $C'=\cup C$.

Then $Q \subseteq C'$ since $\forall E \in C'$, $Q \subseteq E$.

$C' \subseteq B$ since $\forall E \in C'$, $E \subseteq B$.

C' has f.i.p.: Consider $\{D_1,\ldots D_m\} \subseteq C'$

$\exists E_0 \subseteq C'$ such that $\{D_1,\ldots D_m\} \subseteq E_0$

for otherwise $\exists D_i$ such that $\forall E$, $D_i \notin E$ and $D_i \notin C'$.

Since E_0 has f.i.p., $\cap\{D_1,\ldots D_m\} \neq \phi$

Thus $\forall C$ in P, there is an upper bound.

By Zorn's Lemma, P contains a subfamily Q' that is maximal with respect to f.i.p.

Then $Q \subseteq Q' \subseteq B$.

Index Q' by Γ, i.e. let $Q'=\{G_g : g \in \Gamma\}$

Then $\forall g$, $G_g=S_{g1} \cup S_{g2} \cup \ldots \cup S_{gn}$ where $S_{gi} \in S$.

(definition of element of the base for the closed sets)

We want to show that $\forall g$, $\exists S_{gi} \in Q'$.

Suppose not, i.e. $\exists G_g$ such that $\forall i$, $S_{gi} \notin Q'$.

For each i, consider $Q'(g_i)=Q' \cup \{S_{gi}\}$.

Since Q' is maximal with respect to f.i.p., $Q'(g_i)$ does not have f.i.p.

Then $\exists H_i \subseteq Q'$ such that $(\cap H_i) \cap S_{gi}=\phi$. ($S_{gi}$ must be included, for otherwise $\cap H_i \neq \phi$ since Q' has f.i.p.)

Let $H=H_1 \cup \ldots \cup H_n \cup \{G_g\}$.

Then $H \subseteq Q'$, H:finite => $\cap H \neq \phi$ (since Q' has f.i.p.)

$p \epsilon \cap H$ => $p \epsilon G_g$ => $\exists S_{gi}$ such that $p \epsilon S_{gi}$.

Then $p \epsilon (\cap H_i) \cap S_{gi} \neq \phi$.

Thus, $\forall g$, $\exists S_{gi} \epsilon Q'$.

$\forall g$, let $S_g \epsilon Q'$ (by the Axiom of Choice for finite sets where $S_g \epsilon \{S_{gi}\}$)

Then $S' = \{S_g : g \epsilon \Gamma\} \subseteq Q'$.

Since Q' has f.i.p., S' has f.i.p.

$S' \subseteq S$, S' has f.i.p. => $\cap S' \neq \phi$.

Since $\forall g$, $S_g \subseteq G_g$, $\cap Q' \neq \phi$.

Since $Q \subseteq Q'$, $\cap Q \neq \phi$.

Thus, $Q \subseteq B$, Q has f.i.p. => $\cap Q \neq \phi$.

By Theorem 1, X is compact. QED[10]

To complete this indirect proof that the Axiom of Choice implies Tychonoff's Theorem, we must get from Alexander's Theorem to Tychonoff's Theorem. As earlier stated, Alexander is only a corollary to Tychonoff, and so we need an application of the full Axiom of Choice to complete the step. We adapt the proof of Alexander => Tychonoff Finite as follows:

When choosing x_a from the set $X_a \setminus U$, we only needed the Axiom of Choice for finite sets since $X_a \setminus U$ was in fact finite. In the full version, X_a is only compact (possibly infinite) so we apply the full Axiom of Choice to get our x_a. Otherwise, the proof

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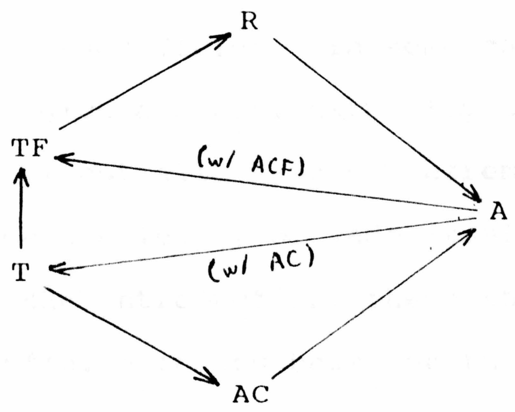
[10] Kelley, General Topology, p.139

follows exactly as written.

This completes the study of the more theoretic aspect of Rado's Selection Principle, and the series of implications that we have arrived at can be summarized in Figure 1 below.



R = Rado's Selection Principle
TF = Tychonoff Finite
T = Tychonoff's Theorem
AC = Axiom of Choice
A = Alexander Subbase Theorem
ACF = Axiom of Choice for Finite Sets

Figure 1.

Chapter 3: Applications

Rado's Selection Principle has applications in branches of mathematics ranging from graph theory to group theory to logic. The remainder of this paper will be devoted to proving six theorems in these and other fields. In some cases, these theorems have proofs which are more naturally contained within their fields, but in almost all, Rado's Theorem provides simpler (after discounting the complexity of Rado itself) ones.

As indicated in the introduction, these theorems yield results concerning infinite spaces based on the characteristics of the finite subspaces. Frequently, that the proposition holds for the finite subspaces is a theorem in itself (e.g. Hall's Theorem); however, our purpose here is not to demonstrate these results (although references will be made in the notes), but rather to extend them to more general situations. Further, the general theorems on infinite spaces are usually stated as double implications, but in going from the infinite to the finite, the step is trivial because of inherited characteristics and so it will be omitted from closer scrutiny.

The first proof has of course already been done in the introduction: the Marriage Problem was a very natural application of Rado in its own field, combinatorics. The first new application is a fairly straightforward one from graph theory. It is the Erdös-deBruijn Theorem which says that a graph is k-colorable if and only if every finite subgraph is k-colorable. We supply the relevant definitions:

Definition. A graph G is a set V of elements called vertices together with a set E of two-element subsets of V called the edges, written $G=(V,E)$. Set theoretically then, pairs of vertices can be connected by at most one edge.

Graphs can be represented pictorially as a set of points (the vertices) in a plane and a set of line segments (the edges) connecting them. Intersections of line segments other than at vertices of V are disregarded. Some graphs cannot be drawn in the plane without these extra intersections.

Definition. Given a graph $G=(V,E)$ and a subset $V' \subseteq V$, the induced subgraph is the graph $G'=(V',E')$ where $\{a,b\} \in E'$ if and only if $a,b \in V'$ and $\{a,b\} \in E$. That is, all possible edges are inherited.

Definition. In a graph $G=(V,E)$, two vertices $a,b \in V$ are said to be adjacent if and only if $\{a,b\} \in E$.

Definition. A coloring of a graph $G=(V,E)$ is a function $f:V \to C$, where C is any set, the colors, such that if $a,b \in V$ and $\{a,b\} \in E$ then $f(a) \neq f(b)$. That is, adjacent vertices are assigned different colors. If $|\text{range}(f)|=k$, then f is called a k-coloring. The graph G is said to be k-colorable if there exists a coloring f such that $|\text{range}(f)|\leq k$.

Theorem. (Erdös-deBruijn) A graph is k-colorable if and only if every finite subgraph is k-colorable.

Proof: Let $G=(V,E)$ be an infinite graph. (The theorem is trivial otherwise.) Let $\hat{J}$ be the class of all finite subsets of V. $\forall B\in\hat{J}$, let G_B be the induced subgraph.

Then, by hypothesis, G_B is k-colorable, say by the function $f_B:B\to C$, where $|C|=k$. Since every finite subgraph is k-colorable, we may use C as the range space for all local choice functions f_B.

Then, by Rado, there exists a global choice function $f:V\to C$ such that given $B\in\hat{J}$, $\exists D\in\hat{J}$, $D\supseteq B$, such that $f(b)=f_D(b)$ $\forall b\in B$. We must show that f is a k-coloring; that is, we must show that any two adjacent vertices are differently colored.

Consider any two adjacent points $a,b\in V$, and let $B=\{a,b\}$. Let $D\in\hat{J}$, $D\supseteq B$ be the set whose existence is guaranteed by Rado. Then, $f(a)=f_D(a)\neq f_D(b)=f(b)$. Thus, f is a k-coloring.

QED[11]

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[11] L. Mirsky and Hazel Perfect, "Systems of Representatives," Journal of Mathematical Analysis and Applications, v.15 (1966), p.541

Another graph theory theorem, due to Wolk, gives the orientability of a graph in terms of the orientability of its finite subgraphs, but first we must explain what an orientation is:

Definition. An oriented graph $G=(V,E')$ is a collection of vetices V and a collection E' of ordered pairs of distinct vertices such that if $(a,b)\in E'$ then $(b,a)\notin E'$. Note that every edge must be oriented for the graph to be oriented.

Definition. The relation T defined by $aTb \iff (a,b)\in E'$ is called the orientation of G. This relation is said to be a transitive orientation if and only if $aTb, bTc \Rightarrow aTc$. For an unoriented graph $G=(V,E)$ to admit a transitive orientation means that there exists such a relation T such that if $\{a,b\}\in E$, then either aTb or bTa. Again, every edge must be oriented consistent with transitivity.

Example.

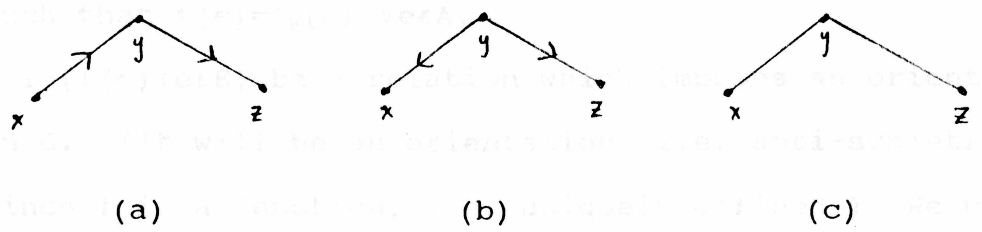

Figure 2.

In Figure 2, the graph (a) is not transitively oriented because $E'=\{(x,y),\ (y,z)\}$ implies xTy and yTz, but it is not the case that xTz because $(x,z)\notin E'$. The graph (b) is transitively oriented because the transitivity hypothesis is null. Thus, the unoriented graph (c) admits a transitive orientation, e.g. (b).

<u>Theorem</u>. (Wolk) If every finite subgraph of an unoriented graph $G=(V,E)$ admits a transitive orientation, then so does G.

<u>Proof</u>: Since the object of the proof is to create an orientation, the obvious choice to be made is the selection of an orientation for each edge. Thus, if for each $e=\{x,y\}\in E$, we let $\Gamma_e=\{(x,y),(y,x)\}$, then E is isomorphic to $\{\Gamma_e:e\in E\}$, our infinite family of finite sets.

By hypothesis, $\forall A\subseteq E$, A:finite, we may define f_A, a local choice function on $\{\Gamma_e:e\in A\}\simeq A$, such that the relation image(f_A) imposes a transitive orientation on the edges. (For notational convenience, let $f_A(e)=f_A(\Gamma_e)$.)

Then by Rado, there exists a global choice funtion f on E such that given $E\supseteq A$:finite, there exists $B\supseteq A$, $E\supseteq B$:finite such that $f(e)=f_B(e)$ $\forall e\in A$.

Let $T=\{f(e):e\in E\}$ be a relation which imposes an orientation on G. (It will be an orientation, i.e. anti-symmetric, since f is a function, i.e. uniquely defined.) We must show that T is transitive:

Suppose xTy and yTz. Let $A=\{\{x,y\},\{y,z\}\}$.

Then $\exists B \supseteq A$ such that $f_B(\{x,y\})=f(\{x,y\})=(x,y)$

$f_B(\{y,z\})=f(\{y,z\})=(y,z)$.

But the relation image(f_B) imposes a transitive orientation on the subgraph $G_B=(V_B,B)$, so $(x,z)\in$ image(f_B).

Then $f_B(\{x,z\})=(x,z)$ and $\{x,z\}\in B\subseteq E$.

With the existence of $\{x,z\}\in E$, let $A'=\{\{x,y\},\{y,z\},\{x,z\}\}$

Then $\exists B'\subseteq A'$ such that $f_{B'}(\{x,y\})=f(\{x,y\})=(x,y)$

$f_{B'}(\{y,z\})=f(\{y,z\})=(y,z)$

Also, $f_{B'}(\{x,z\})=(x,z)$ by transitivity, and since $\{x,z\}\in A'$

$f(\{x,z\})=f_{B'}(\{x,z\})=(x,z)$.

Thus, xTz and T is a transitive relation imposing a transitive orientation on G. QED[12]

The next application of Rado, this one to a theorem of B.H. Neumann, concerns the ordering of infinite groups. It is very similar to the latter in that it builds an ordering of the whole structure by choosing between the two possible orderings of any particular pair of elements. Thus, once again the finite sets in our infinite family will be two-element sets corresponding to these two possible local orderings. Let us be more precise about what is meant by an ordering of a set:

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[12] E.S. Wolk, "A Note On 'The Comparability Graph of a Tree'," <u>Proceedings of the AMS</u>, v.16 (1965), p.18

Definition. A binary relation $<$ is said to be an ordering on a set S if and only if the following conditions are satisfied:

1) (total, anti-symmetric) Given $a,b \in S$ distinct, either $a<b$ or $b<a$ but not both.

2) (transitive) Given $a,b,c \in S$, if $a<b$ and $b<c$, then $a<c$.

Further, if S is a group, then the ordering relation is said to be compatible if it is preserved under the group operation:

3) Given $a,b,c \in S$, if $a<b$, then $ac<bc$ and $ca<cb$.

Theorem. (Neumann) An infinite group G has a compatible order if and only if every finitely generated subgroup of G has a compatible order.

Proof: Let G have compatible orders for all its finitely generated subgroups.

Let $I=\{H \subseteq G: |H|=2\}$. Let $W=\{(a,b): a,b \in G, a \neq b\}$

For all $H \in I$, if $H=\{a,b\}$, let $\Gamma_H=\{(a,b),(b,a)\} \subseteq W$.

Let $\hat{J}$ be the class of all finite subsets of I.

Let $J \in \hat{J}$ and let $H_J = \cup\{H: H \in J\} \subseteq G$.

Let G_J be the subgroup generated by H_J.

Then, by hypothesis, there exists $<_J$, a compatible ordering of G_J.

Define f_J on $\{\Gamma_H: H \in J\}$ by:

If $H=\{a,b\}$, let $f_J(H)= \begin{array}{l} (a,b) \text{ if } a<_J b \\ (b,a) \text{ if } b<_J a. \end{array}$

By Rado, there exists $f:\{\Gamma_H: H \in I\} \rightarrow W$, a global choice function, such that $\forall J \in \hat{J}$, $\exists K \in \hat{J}$, $J \subseteq K$:finite and

$f(H)=f_K(H)\ \forall H\epsilon J$.

Define < on G by a<b if and only if $f(\{a,b\})=(a,b)$.

We must show that this is a compatible ordering:

(1) Since f is global, < is total.

Since f is a function, < is anti-symmetric.

(2) Suppose a<b, b<c. Then consider $J=\{\{a,b\},\{b,c\},\{a,c\}\}$.

$\exists K\supseteq J$ such that $f(\{a,b\})=f_K(\{a,b\})=(a,b) \Rightarrow a<_K b$

$f(\{b,c\})=f_K(\{b,c\})=(b,c) \Rightarrow b<_K c$.

By transitivity of $<_K$, $a<_K c \Rightarrow (a,c)=f_K(\{a,c\})=f(\{a,c\})$

Thus, a<c.

(3) Let $x,y,z\epsilon G$ and suppose x<y.

Consider $J=\{\{x,y\},\{xz,yz\},\{zx,zy\}\}$.

$\exists K\supseteq J$ such that $f(\{x,y\})=f_K(\{x,y\})=(x,y) \Rightarrow x<_K y$

(without loss of generality)

Then by preservation under group structure,

$xz<_K yz \Rightarrow (xz,yz)=f_K(\{xz,yz\})=f(\{xz,yz\})$.

Thus, xz<yz and similarly zx<zy.

Thus, < is a compatible order on G. QED[13]

Our final three results are from three different fields, each with its own axioms and terminology. The first is Dilworth's Theorem from set theory which relates incomparable elements to disjoint chains in infinite partially ordered sets. Clearly, though, we need some background:

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[13] Mirsky and Perfect, p.541.

Definition. A partially ordered set (poset) S is a collection of elements together with a binary relation $\leq$ on those elements satisfying:

(1) (reflexivity) $\forall x \epsilon S \quad x \leq x$

(2) (anti-symmetry) $\forall x,y \epsilon S \quad x \leq y$ and $y \leq x \Rightarrow x=y$

(3) (transitivity) $\forall x,y,z \epsilon S \quad x \leq y$ and $y \leq z \Rightarrow x \leq z$

Definition. Two elements $x,y \epsilon S$:poset with relation $\leq$ are said to be comparable if and only if $x \leq y$ or $y \leq x$. If neither holds, then x and y are said to be incomparable.

Definition. A chain is a subset $P \subseteq S$:poset such that any two elements of P are comparable.

Definition. A poset S can be decomposed into chains if there exist disjoint chains $\{Q_i\}$ such that $\forall x \epsilon S$, $\exists i$ such that $x \epsilon Q_i$.

Theorem. (Dilworth) The maximum number of pairwise incomparable elements in an infinite partially ordered set P is equal to the minimum number of pairwise disjoint chains into which P can be decomposed, if these numbers are finite.

Proof: Assume the theorem is true for finite posets.

Suppose P is an infinite poset and that any subset of P with more than k elements has at least two comparable elements. (That is, suppose k is the maximum number of pairwise incomparable elements.)

For $x \epsilon P$, let $\Gamma_x = \{1,2,\ldots k\} = \Gamma$. Thus, our family of finite sets will be Γ for each $x \epsilon P$.

For any $Q \subseteq P$, Q:finite, $\exists \{Q_i : 1 \leq i \leq k\}$: pairwise disjoint chains (some of which may be empty) such that $Q = Q_1 \cup \ldots \cup Q_k$. This is true by our initial supposition.

Then for all such Q, define $f_Q : Q \to \Gamma$, our local choice functions, by $f_Q(x) = i$ if and only if $x \epsilon Q_i$ in the given decomposition. Thus, if $x, x' \epsilon Q$ and $f_Q(x) = f_Q(x') = i$, then $x, x' \epsilon Q_i$ and x, x' are comparable.

By Rado's Selection Principle, there exists a global choice function $f : P \to \Gamma$ such that $\forall Q \subseteq P$, Q:finite, $\exists R \supseteq Q$, R:finite such that $f(x) = f_R(x)$ for all $x \epsilon Q$.

Define $P_i = \{x \epsilon P \mid f(x) = i\}$, $1 \leq i \leq k$.

We must show that $\{P_i : 1 \leq i \leq k\}$ decomposes P into disjoint chains.

$P = P_1 \cup \ldots \cup P_k$ since given $x \epsilon P$, $f(x) \epsilon \Gamma$ and thus x is in at least one of the P_i.

$P_i \cap P_j = \phi$, $i \neq j$, since if $x \epsilon P_i \cap P_j$, then $f(x) = i$ and $f(x) = j$, contradicting the fact that f is a function. Thus, each x is in at most one of the P_i.

Thus, $P_1 \cup \ldots \cup P_k$ is a disjoint decomposition of P.

It remains to be shown that each P_i is a chain.

Take $Q = \{x, x'\} \subseteq P$ and suppose $f(x) = f(x')$. (i.e. x, x' are in the same chain)

Then $\exists R \supseteq Q$, R:finite such that $f_R(x) = f(x)$ and $f_R(x') = f(x')$.

Thus, $f_R(x) = f_R(x')$ and so x, x' are comparable. QED

Logic is that branch of mathematics that tries to determine the truth of a given statement based solely on the structure of the statement and the truth or falsity of the components of the structure; that is, without regard for the meaning of the statement or its components. The propositional calculus is the formal logical system L consisting of:

1) the propositional letters $X=\{p_i : i\in I\}$ which are the components or building blocks of the structure and which may variably be either true or false.

2) the connectives $\neg$ (negation) and $\rightarrow$ (implication) (in L these are the only two connectives because they are sufficient for generating all possible truth tables, but in general there may be more), and the parentheses (and) which are the structure into which the propositional letters are put, creating:

3) the set of propositions W governed by the rules of structure: (i) $p_i \in W \ \forall i \in I$

   (ii) if $A,B \in W$, then $(\neg A) \in W$ and $(A\rightarrow B) \in W$

4) the axioms: (i) $(A\rightarrow(B\rightarrow A))$

   (ii) $((A\rightarrow(B\rightarrow C))\rightarrow((A\rightarrow B)\rightarrow(A\rightarrow C)))$

   (iii) $(((\neg A)\rightarrow(\neg B))\rightarrow(B\rightarrow A))$

   which establish these propositions as true regardless of the assignment of truth or falsity to A,B and C.

5) the rule(s) of deduction (in L there is just one):

   Modus Ponens: if A is true, and

   if $(A\rightarrow B)$ is true, then B is true.

Given a set of propositional letters $X=\{p_i:i\in I\}$ where $|I|$ is any cardinal, the set of propositions generated by using the rules of structure and the connectives ad infinitum will necessarily be infinite. Two (perhaps structurally different) propositions, however, may be logically equivalent in that they always take on identical truth values independent of the truth values of their propositional letters, as determined by the rules of deduction or by truth tables (more about these later). Thus, the set of propositions may be divided into equivalence classes, and in fact, if $|I|$ is finite, the number of equivalence classes will be $2^{2^{|I|}}$. One more formal definition gives the necessary machinery for our next theorem:

<u>Definition</u>. A <u>valuation</u> $v:W\to\{T,F\}$ is a map which assigns to each proposition (where W is the set of propositions) the value either T (true) or F (false) such that:

1) $v(A)\neq v(\neg A)$
2) $v(A\to B)=F$ if and only if $v(A)=T$ and $v(B)=F$.

These conditions insure that if two propositions are logically equivalent, they will be given the same truth value.

We note for emphasis that since $X\subseteq W$ (by rule of structure #1), a valuation also assigns truth values to the propositional letters. We note further that since the set of propositions is generated by the set of propositional letters, a restriction of the valuation to the latter set can be uniquely extended by truth tables which exhaustively apply conditions 1 and 2 from the definition of valuation until all propositions have been assigned truth values.

The groundwork is now complete enough to prove the Compactness Theorem for the Propositional Calculus using Rado. Although this result has already been established using principles from just logic theory, to the best of our knowledge, the following is a new proof.

<u>Theorem</u>. (The Compactness Theorem for the Propositional Calculus)

For a (possibly infinite) set of propositions Σ, there exists a valuation v such that $v(s)=T$ for all $s \in \Sigma$ provided the same is true for all finite subsets of Σ.

<u>Proof</u>: Let Σ be such a set of propositions (i.e. with the finite subset property).

Let $X=\{p \mid \exists s \in \Sigma$ such that p is a propositional letter in s$\}$

For each $A \subseteq X$, A:finite:

Let W_A be the set of propositions generated by A (using rules of structure 1 and 2).

Let $\Sigma_A=\Sigma \cap W_A$.

Let Σ_A' be the set of equivalence classes of Σ_A. ($|\Sigma_A|$ is finite! and has size at most $2^{2^{|A|}}$.)

Let $\Sigma_A''=\{s \mid \exists [s] \in \Sigma_A'$ such that $s \in [s]$ and $s \neq t \Rightarrow [s] \neq [t]\}$.

(These sets are not uniquely determined, but we do not need any version of the Axiom of Choice to create them: since propositions are of finite (integral) length, including all connectives, we may (for example) use well-ordering to select the smallest. If this is still not unique, we may place an ordering on A, etc.)

Then by hypothesis, for Σ_A" there exists $v_A: W_A \to \{T, F\}$, a local choice function on A (and a valuation on W_A by truth table extension) such that $v_A(s)=T \quad \forall s \in \Sigma_A$".

Define such a v_A for each $A \subseteq X$, A:finite.

Then, by Rado's Selection Principle, $\exists v: W_X \to \{T, F\}$, a global choice funtion on X (and a valuation on W_X by truth table extension) such that for $A \subseteq X$, A:finite, $\exists B \supseteq A$, B:finite such that $v(p)=v_B(p) \quad \forall p \in A$.

Consider $s \in \Sigma$ and its set of propositional letters $X_{\{s\}} \subseteq X$.

Then $\exists B \supseteq X_{\{s\}}$, B:finite such that $v(p)=v_B(p) \quad \forall p \in X_{\{s\}}$.

Then find (as previously selected) $t \in \Sigma_B$" such that $s, t \in [s] \in \Sigma_B'$.

Then $v(s)=v_B(s)$ (since the truth value of s depends only on the truth values on $X_{\{s\}}$)

$=v_B(t)$ (since $s, t \in [s]$, i.e. are equivalent)

$=T$ ($t \in \Sigma_B$").

Thus, $\forall s \in \Sigma$, $v(s)=T$. QED

The last result is an extension of Landau's theorem from tournament theory which gives the conditions under which a score vector has a tournament corresponding to it. We give one more set of definitions:

__Definition__. A __tournament__ T on a set N is a matrix of 0's and 1's

$$T=\{t(x,y)\,|\,x,y\in N,\ t(x,y)\in\{0,1\}\}$$

where: 1) $t(x,x)=0\ \forall x\in N$ and

2) $t(x,y)+t(y,x)=1$ if $x\neq y$.

Thus, a mathematical tournament can be thought of as the score-sheet of an everyday tournament wherein every team plays every other team and the winner scores 1 with no draws.

__Definition__. The x-th __row sum__ or __score__ of a tournament T is given by $r_x=|\{y\in N:t(x,y)=1\}|$. The vector $r=(r_i:i\in N)$ is called the __score vector__ of the tournament.

__Definition__. A tournament is said to be __row-finite__ if and only if each of its row sums is finite.

__Definition__. A vector $r=(r_i:i\in N)$ is said to satisfy the __Landau condition__ if and only if $\forall A\subseteq N$, A:finite, $\Sigma_{i\in A}r_i\geq\binom{|A|}{2}$.

In the following, let L={r:r satisfies the Landau condition}.

Then for finite tournaments (i.e. N is finite), we get Landau's Theorem which says[14]:

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[14] Bang and Sharp, "Score Vectors on Tournaments," __Journal of Combinatorial Theory, Series B__, v.26, Number 1 (Feb. 1979), p.83

Theorem. (Landau) Given a finite set N and a vector $r=(r_i : i \in N)$ of non-negative integers, there exists a tournament on N with score vector r if and only if: 1) $r \in L$

2) $\Sigma_{i \in N} r_i = \binom{|N|}{2}$.

(In our everyday model, the second condition guarantees that the correct number of games is played.)

As usual, we will not give the proof here. Our aim is to extend Landau's Theorem (or some altered version of it) using Rado so that it includes infinite tournaments. For an infinite tournament, the second condition is ambiguous at best, and it turns out that only the first condition is needed. However, in the absence of the second, and in order to apply Landau's Theorem to our local (i.e. finite) choice functions (which will generate tournaments), we need the following lemma:

Lemma. Given a finite set N and a vector $r=(r_i : i \in N)$ of non-negative integers such that $r \in L$ but $\Sigma_{i \in N} r_i > \binom{|N|}{2}$, there exists a vector $t=(t_i : i \in N)$ such that (1) $t \in L$,

(2) $\forall i \in N$, $t_i \leq r_i$, and

(3) $\Sigma_{i \in N} t_i = \binom{|N|}{2}$.

Proof: Suppose $r \in L$ as above.

(*) Let $r_j = \max\{r_i : i \in N\}$. (if not unique, choose smallest j)

Let $r_j' = r_j - 1$,

$r_i' = r_i$, $i \neq j$.

Claim $r'=(r_i':i\epsilon N) \epsilon L$. If we can show this to be true, we may repeat process (*) finitely many times to get our result.

Proof of Claim: Notice first $1+\Sigma_{i\epsilon N}r_i'=\Sigma_{i\epsilon N}r_i>\binom{|N|}{2}$

$$\Rightarrow \quad \Sigma_{i\epsilon N}r_i'\geq\binom{|N|}{2}.$$

Suppose $r'\notin L$. Then $\exists A\subseteq N$, $j\epsilon A$, with $|A|=a$, such that

$$\Sigma_{i\epsilon A}r_i'<\binom{a}{2}$$

$$\text{but} \quad \Sigma_{i\epsilon A}r_i \geq\binom{a}{2} \quad \text{by hypothesis}$$

$$\text{thus} \quad \Sigma_{i\epsilon A}r_i =\binom{a}{2}.$$

Choose $k\notin A$. Let $A'=(A\cup\{k\})\setminus\{j\}$.

If $r_k\leq r_j-1$, then $\Sigma_{i\epsilon A'}r_i=\binom{a}{2}-1<\binom{a}{2}$. Then $r\notin L$, contradicting the hypothesis. Thus, $r_k=r_j$.

Then consider, (1) $r_k+\Sigma_{i\epsilon A}r_j \geq \binom{a+1}{2}$

$$r_k \geq \binom{a+1}{2}-\binom{a}{2}$$

$$= a$$

(2) Since $\Sigma_{i\epsilon A\setminus\{j\}}r_i \geq \binom{a+1}{2}$,

$$r_j = \Sigma_{i\epsilon A}r_i - \Sigma_{i\epsilon A\setminus\{j\}}r_i$$

$$\leq \binom{a}{2}-\binom{a-1}{2}$$

$$= a-1$$

Thus $r_j \leq a-1 < a \leq r_k$, contradicting the maximality of r_j. Thus $r'\epsilon L$ and we may repeat process (*) until we get our new vector t. QED

We may now prove Landau's Theorem for infinite tournaments. This too is a new proof to the best of our knowledge.

Theorem. Given a finite set N and a vector $R=(r_i:i\epsilon N)$ of non-negative integers such that $r\epsilon L$, there exists a row-finite tournament T on N with score vector r.

Proof: Let $I=\{A\subset N:|A|=2\}$.

For each $A\epsilon I$, $A=\{x,y\}$, let $\Gamma_A=\{(x,y),(y,x)\}$.

Consider any $J\subset I$, J:finite.

Define $N_J=\cup\{A:A\epsilon J\}$

$=\{n_1,\ldots,n_j\}\subset N$

Then let $r_J=(r_{n_1},\ldots,r_{n_j})$.

Then since r_J inherits the Landau condition from r, by the lemma, $\exists s_J=(s_{n_1},\ldots,s_{n_j})$ with $s_{n_k}\le r_{n_k}$ $\forall k$ such that $\Sigma_{i\epsilon N_J}s_i=\binom{|N_J|}{2}$.

By Landau's Theorem, there exists a tournament T_J on N_J with score vector s_J.

Define $f_J:J\to\cup\{\Gamma_A:A\epsilon J\}$ by $f_J(\{x,y\})=$ (x,y) if $t_J(x,y)=1$

(y,x) if $t_J(y,x)=1$

By Rado, $\exists f:I\to\cup\{\Gamma_A:A\epsilon I\}$ such that $\forall J\subset I$, J:finite, $\exists K\supseteq J$, K:finite, such that $f(A)=f_K(A)$ $\forall A\epsilon J$.

Define a tournament T on N by:

$t(x,y)=1$ and $t(y,x)=0$ if $f(\{x,y\})=(x,y)$ and

$t(x,y)=0$ and $t(y,x)=1$ if $f(\{x,y\})=(y,x)$ and

$t(x,x)=0$ $\forall x\epsilon N$.

We must show that T is row-finite.

Suppose the i-th row sum is $>r_i$ or infinite. Then let τ be the cardinal such that $t_{i\tau}$ is the $(r+1)$-st 1 in the i-th row. Then let $J=\{\{x,y\}:x,y\le\tau\}$. By Rado, $\exists K\supseteq J$, K:finite such that $f(\{x,y\})=f_K(\{x,y\})$ $\forall\{x,y\}\epsilon J$. Then,

$$\Sigma_{k\in N_K} t_K(i,k) = \Sigma_{k\in N_K} t(i,k)$$
$$\geq \Sigma_{k\in N_J} t(i,k)$$
$$= r_i+1$$

which contradicts the manner in which T_K was created.

Thus $s_i \leq r_i \quad \forall i$.

Thus T's score vector s is term by term less than r. Using Bang and Sharp's method for inductively augmenting the row sums,[15] we may produce a new tournament T' with score vector precisely r. QED

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[15] Bang and Sharp, p.83

Rado's Selection Principle is an important tool for extending results on finite structures to apply to similar infinite structures. Many of these results fit under the large heading of compactness which in general says just that - that infinite collections with certain properties have finite subcollections with those same properties. In particular, we have used two explicit definitions of compactness in restricted settings: topological compactness in Tychonoff's Theorem and compactness of sets of propositions in logic.

As stated earlier, Rado's Selection Principle is weaker than the Axiom of Choice and its equivalent theorems; Rado may be used only on families of **finite** sets. For instance, here are two other theorems for which Rado is inapplicable. The first is a theorem from algebra which says that any integral domain can be embedded in a field. If we were to try to use Rado to prove this, we might define the range of our choice functions to be all possible fields or all possible embeddings. Each of these sets, however, is clearly infinite, and Rado does not help. Second, the Compactness Theorem for the First-Order Predicate Calculus from mathematical logic says that any first-order system has a model in which all of its propositions are true. At first glance, this seems to be fertile ground for Rado, but a possible application would involve the set of models as the range of the choice functions, and this set is not even countable. Thus, Rado is not the tool we need in these situations. Only the full Axiom of Choice is applicable in these situations.

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