

# WHICH WILL SURVIVE?

## An Investigation into the Modeling of Complex Ecosystems

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### Abstract

When modeling the population of a species, the logistic equation is the first that is invoked; however, this model has some shortcomings when predicting actual population change. This has led to the search of a model that will be able to help predict population change using the current technology, which means that more population-affecting parameters, such as space, density, altitude and interaction, could be included into the model. When adding additional parameters, the complexity of the model is increased dramatically; thus we encounter another problem: not having technology that is able to solve the model. Therefore, one needs to be able to find a model that considers additional parameters while maintaining the integrity of the computer system. This paper will summarize the results that were obtained and possible future directions that could be explored.

## 1 INTRODUCTION

This past summer, I participated in a Research Experience for Undergraduates Program at Texas A & M University. Under the supervision of Dr. Jay Walton and Paulo Lima-Fihlo, I worked with two other students, one graduate and one other undergraduate. There, we studied previous population models and their shortcomings. Dr. Walton gave us the problem of incorporating different parameters into a model, mainly height and migration. During the REU, we wrote computer programs using MATLAB that would graph these population models. When I returned to Washington and Lee, I decided to focus more on the mathematics behind the models and did so under the supervision of Dr. Alan McRae and Dr. Michael Evans. The following paper is a synopsis of this research.

Species interaction affects the population dynamics of all species involved. A trophic web, or a web of interacting species, exists and makes structurally complex communities. In this paper, we will focus on two-species systems that

are competitive in nature. In competitive interaction, two or more species compete for the same limited resource, hence inhibiting each others' growth. Thus, the question lies in whether these species can coexist or will one species become extinct? By using mathematical modeling, one hopes to predict how the populations will change over time. Unfortunately, there are many parameters which if not taken into account make the model unrealistic. This has been the case with many of the simpler models historically applied to this problem. In this project, we shall attempt to improve the model by incorporating two additional parameters: space and migration

## 1.1 BACKGROUND

As early as 1925, mathematicians have been trying to model population growth. Two of the first were Lotka and Volterra. The Lotka-Volterra two-species competition model is a simple model where each species has logistic growth in the absence of the other. The model is the following:

$$\begin{cases} \dot{x} = x(t) f(x(t), y(t)) \\ \dot{y} = y(t) g(x(t), y(t)). \end{cases}$$

Here  $\dot{x}$  and  $\dot{y}$  are the growth rates of the two species,  $x(t)$  and  $y(t)$  are the densities of the two species, and  $f(x(t), y(t))$  and  $g(x(t), y(t))$  are functions of the two species' birth and death rates which incorporate their carrying capacities. What this model does not account for is the space that the species live in and their migration. These parameters are what we shall look into.

### 1.1.1 Underlying Definitions for Problem One: Space

When talking about a species, it is common to discuss its migration patterns. Obviously, a species can only travel a certain distance in one day and might not be able to live beyond a certain elevation. Thus, knowing the terrain that the species resides in is very important. The problem is that when looking at a plot of space, the length between two points can be deceiving since the three-dimensional world is being projected onto a flat surface, for example a piece of paper. Thus, we lose the depth, or height, of the area in question; this raises problems with the migration of the species. Hence, we must be able to find the path on the surface where the species can live.

**Definition 1.1.1.** *A surface  $S \subset \mathbb{R}^3$  is regular if every point in  $S$  has an open set with a coordinate chart in  $\mathbb{R}^2$ , where a coordinate chart is a way of expressing the points of a small neighborhood, usually on a manifold, as coordinates in Euclidean space. In our case, the coordinate chart is a function that maps a set in  $S$  to  $\mathbb{R}^2$ .*

Next, we need to define a tangent vector and tangent plane.

**Definition 1.1.2.** A tangent vector,  $\vec{v}$ , to the regular surface  $S \subset \mathbb{R}^3$  at point  $p \in S$  is a velocity vector of a curve  $\alpha : (-\epsilon, \epsilon) \rightarrow S$  satisfying  $\alpha(0) = p$  and  $\alpha'(0) = \vec{v}$ .

The collection of all tangent vectors to  $S$  at  $p \in S$  is called the tangent plane to  $S$  at  $p$ , denoted  $T_p S$ .

Knowing this, we can define the First Fundamental Form, which will be needed later.

**Definition 1.1.3.** The First Fundamental Form,  $I_p : T_p S \rightarrow \mathbb{R}$ , is defined by

$$I_p(\vec{v}) = \vec{v} \cdot \vec{v} = \|\vec{v}\|^2 \geq 0.$$

We can specialize the previous definitions to incorporate a height function into our model by doing the following: in local coordinates, let  $\mathbb{X} : U \subset \mathbb{R}^2 \rightarrow S$ , then  $\vec{v} \in T_p S$  is written as

$$\vec{v} = u' \vec{\mathbb{X}}_u + v' \vec{\mathbb{X}}_v.$$

Hence,

$$\begin{aligned} I_p(\vec{v}) &= (u' \vec{\mathbb{X}}_u + v' \vec{\mathbb{X}}_v) \cdot (u' \vec{\mathbb{X}}_u + v' \vec{\mathbb{X}}_v) \\ &= (\vec{\mathbb{X}}_u \cdot \vec{\mathbb{X}}_u)(u')^2 + 2(\vec{\mathbb{X}}_u \cdot \vec{\mathbb{X}}_v)u'v' + (\vec{\mathbb{X}}_v \cdot \vec{\mathbb{X}}_v)(v')^2 \\ &\equiv \mathbb{E}(u')^2 + 2\mathbb{F}(u', v') + \mathbb{G}(v')^2. \end{aligned} \quad (1)$$

In our case,  $\mathbb{X} : U \rightarrow S$ , where  $U$  is the unit square and  $\mathbb{X}$  is given by  $(u, v) \rightarrow (u, v, h(u, v))$ , where  $h(u, v)$  is a function of  $u$  and  $v$  that pertains to the height of the surface. Thus, by equation 1 above, we see that  $\mathbb{E} = 1 + (h_u(u, v))^2$ ,  $\mathbb{F} = h_u(u, v)h_v(u, v)$ , and  $\mathbb{G} = 1 + (h_v(u, v))^2$ .

Now that we have a function that can give us the height of a surface, we need to know the length of a given section of the surface in order to model how far a species can travel on a given terrain.

**Definition 1.1.4.** Given  $\alpha : [a, b] \rightarrow S$ , where  $S$  is a smooth surface, then its length is given by

$$\begin{aligned} L(\alpha) &= \int_a^b |\alpha'(t)| dt \\ &= \int_a^b \sqrt{I_{\alpha(t)}(\alpha'(t))} dt. \end{aligned}$$

This is our function that defines length; hence, if we know the maximum elevation that a species can survive on and the distance it can travel per unit time, we will be able to model that species' interaction more accurately. We can now incorporate our height and length functions into the model, but first we must also be able to model the migration patterns of the species.

### 1.1.2 Underlying Definitions of Problem Two: Migration

Obviously, we need to take into account the fact that species will move throughout their terrain. In order to model this, we must first go back to Multivariable Calculus.

**Definition 1.1.5.** The gradient vector, denoted  $\nabla f$ , of a differential function,  $f(x, y, z)$ , at the point  $(a, b, c)$  is given by,

$$\text{grad}(f(a, b, c)) = f_x(a, b, c)\vec{i} + f_y(a, b, c)\vec{j} + f_z(a, b, c)\vec{k}.$$

Once we have the gradient of our function, we can use it to model migration, but first we need to know the following from Linear Algebra:

**Definition 1.1.6.** If  $A$  is a  $n \times n$  matrix, then the trace of  $A$  is the summation of the principle diagonal entries, or  $\text{trace}(A) = a_{11} + a_{22} + \dots + a_{nn}$ .

The divergence,  $\text{div}(\vec{W})$ , of a vector field,  $\vec{W}$ , at a point  $p \in S$  is the trace of the linear map,  $D(\vec{W}) : T_p S \rightarrow T_p S$  given by

$$D(\vec{W}) = \text{proj}_{T_p S} \left( \frac{d}{dt} (\vec{W} \cdot \alpha) \Big|_{t=0} \right)$$

where the projection is a linear transformation from one vector space to a subspace of that given vector space.

**Definition 1.1.7.** Given that  $f(p) : S \rightarrow \mathbb{R}$ , where  $p \in S$ , the Laplacian of  $f(p)$ , denoted  $\Delta f(p)$ , is given by the function

$$\begin{aligned} \Delta f(p) &= \text{div}(\text{grad } f) \\ &= \text{div}(\nabla f(p)). \end{aligned}$$

If we use the information from the First Fundamental Form in the previous section and we let  $S$  be the graph of a function  $h : U \rightarrow \mathbb{R}$ , where  $U \subset \mathbb{R}^2$ ,  $f : S \rightarrow \mathbb{R}$  is smooth, and we define  $\mathbb{X} : U \rightarrow \mathbb{R}^3$  as the parameterization  $\mathbb{X}(u, v) = (u, v, h(u, v))$ , then we can prove [1] that

$$\Delta_S f(u, v) = \frac{1}{\delta} \text{div}_U \left( \frac{1}{\delta} B \nabla_U f(u, v) \right), \quad (2)$$

where  $B = \begin{bmatrix} 1 + h_u^2 & -h_u^2 h_v^2 \\ -h_u^2 h_v^2 & 1 + h_v^2 \end{bmatrix}$  is a two-by-two matrix of first order partial derivatives of  $\mathbb{X}(u, v)$  and  $\delta = \sqrt{EG - F^2} = \sqrt{1 + h_u^2 + h_v^2}$ . If we plug  $B$  and  $\delta$  into equation 2 above, we see that [1]

$$\Delta_S f(u, v) = \frac{E f_{uu} + G f_{vv} + 2F f_{uv}}{EG - F^2}.$$

Now that we have an equation for the Laplacian, we hope to be able to account for the movement, or flux, of the species. To do this mathematically, Dr. Walton suggested that we use the Differential Equation of Heat Conduction, or the Heat Balance Equation, as a guide. The energy-balance equation states that the rate of heat entering through a boundary added to the rate of energy generation equals the rate of storage energy. Mathematically:

$$\begin{aligned} \text{Rate of heat entering through the boundary of } V &= - \int_A \vec{q}(\vec{r}, t) \cdot \hat{n} \, dA \\ &= - \int_V \text{div } \vec{q}(\vec{r}, t) \, dv \end{aligned}$$

$$\text{Rate of energy generation} = \int_V g(\vec{r}, t) \, dv$$

$$\text{Rate of storage} = \int_V \rho C_p \frac{\partial T(\vec{r}, t)}{\partial t} \, dv$$

where  $A$  is the surface area of the volume element  $V$ ,  $\hat{n}$  is the normal unit vector to the surface element  $dA$ ,  $\vec{q}(\vec{r}, t)$  is the heat flux vector at  $dA$ ,  $g(\vec{r}, t)$  is the heat generation per unit volume per unit time,  $\rho$  is the density,  $C_p$  is the specific heat, and  $T(\vec{r}, t)$  is the temperature distribution. Since all the rates are being integrated over  $V$ , we see that

$$\int_V \left[ -\text{div } \vec{q}(\vec{r}, t) + g(\vec{r}, t) - \rho C_p \frac{\partial T(\vec{r}, t)}{\partial t} \right] dv = 0.$$

Since the previous equation is derived for an arbitrary small volume element,  $V$ , within the given solid, the volume may be chosen so small that the integral becomes obsolete, thus we can rewrite the equation as

$$-\text{div } \vec{q}(\vec{r}, t) + g(\vec{r}, t) = \rho C_p \frac{\partial T(\vec{r}, t)}{\partial t}.$$

Since the Fourier Law of Heat Conduction states that the heat flux vector equals negative thermal conductivity,  $k$ , multiplied by the temperature gradient, or  $\vec{q}(\vec{r}, t) = -k \nabla T(\vec{r}, t)$ , we can rewrite the equation once again as

$$\text{div } [k \nabla T(\vec{r}, t)] + g(\vec{r}, t) = \rho C_p \frac{\partial T(\vec{r}, t)}{\partial t}.$$

This equation is intended for temperature or space dependent  $k$  and temperature dependent  $C_p$ ; thus, if the thermal conductivity is assumed to be constant, we have the following equation:

## 2.1 Mathematical Modeling of Two Species in a "One Di-

$$\frac{\partial T(\vec{r}, t)}{\partial t} - \alpha \Delta T(\vec{r}, t) - \frac{g(\vec{r}, t)}{\rho C_p} = 0, \quad (3)$$

where the thermal diffusivity is  $\alpha = \frac{k}{\rho C_p}$  [3]. Equation 3 is the heat balance equation that accounts for the temperature diffusion through a given volume. Dr. Walton thought this equation could be altered, specifically by replacing the equation for temperature distribution with one for population density, so that it would mathematically model the migration, or flux, of the species on its terrain. We shall pursue this in the next section.

## 2 MODELING INTERACTION OF SPECIES

Let  $\phi(x, t)$  be the population density of a species on  $S$ ; we will denote  $\phi(x, t)$  as  $\phi$  for the rest of the paper. Taking Equation 3 as our model, we write our differential equation as:

$$\frac{\partial \phi}{\partial t} - \beta \Delta \phi = g(\phi)$$

where  $\beta = \alpha$  and  $g(\phi) = \frac{g(\vec{r}, t)}{\rho C_p}$  in Equation 3, and  $g(\phi)$  is the net birth and death rate. We assume Neumann Boundary Conditions, in other words there is no migration at the boundary; thus  $\nabla_s \phi \cdot X_u = 0$  and  $\phi(u, 0) = \phi_o(u)$  where  $u \in S$ . If we take  $N$  to be the carrying capacity of the species, we define  $g(\phi) = \mu(N - \phi) - \delta\phi$ , where  $\mu$  is the birth rate and  $\delta$  is the death rate. Hence, by using Equation 2 and the equation above, our equation to model a species becomes

$$\begin{aligned} \mu(N - \phi) - \delta\phi &= \frac{\partial \phi}{\partial t} - \beta \Delta_s \phi \\ &= \frac{\partial \phi}{\partial t} - \frac{1}{\delta} \operatorname{div}_s \left( \frac{1}{\delta} B \nabla_s \phi \right) \end{aligned} \quad (4)$$

But, since we are dealing with two species, we must modify the model one last time to get the following system of differential equations:

$$\begin{cases} \frac{\partial \phi_1}{\partial t} - \beta_1 \Delta_s \phi_1 = F_1(\phi_1, \phi_2) \phi_1 \\ \frac{\partial \phi_2}{\partial t} - \beta_2 \Delta_s \phi_2 = F_2(\phi_1, \phi_2) \phi_2 \end{cases} \quad (5)$$

where  $F_1 = \mu_1(N_1 - \phi_1) - \nu_1\phi_2$  and  $F_2 = \mu_2(N_2 - \phi_2) - \nu_2\phi_1$  when  $\mu_i$  is the birth rate of the  $i^{\text{th}}$  species,  $\nu_i$  is the death rate of  $i^{\text{th}}$  species, and  $\phi_i$  is the population density of the  $i^{\text{th}}$  species.

## 2.1 Mathematical Modeling of Two Species in a "One Dimensional" World

We will first assume that our species lives in a "one dimensional" world, let us say the interval  $[a, b]$  on the real line. Since we are now dealing with the one-dimensional case, we can rewrite Equation 5 as

$$\begin{cases} \frac{\partial \phi_1}{\partial t} - \beta_1 \frac{\partial^2 \phi_1}{\partial x^2} = F_1(\phi_1, \phi_2)\phi_1 \\ \frac{\partial \phi_2}{\partial t} - \beta_2 \frac{\partial^2 \phi_2}{\partial x^2} = F_2(\phi_1, \phi_2)\phi_2. \end{cases}$$

To simplify matters, we will let  $\phi_1 = U$  and  $\phi_2 = V$ . Thus, our model for the one dimensional world is the following:

$$\begin{cases} \frac{\partial U}{\partial t} = \beta_1 \frac{\partial^2 U}{\partial x^2} + (\mu_1(N_1 - U) - \nu_1 V)U \\ \frac{\partial V}{\partial t} = \beta_2 \frac{\partial^2 V}{\partial x^2} + (\mu_2(N_2 - V) - \nu_2 U)V. \end{cases} \quad (6)$$

In order to simplify the above equations, we will non-dimensionalize the terms [2]. We will have our new  $t = \mu_1 t$ ,  $u = U$ ,  $v = V$ ,  $\mu = \frac{\mu_2}{\mu_1}$ ,  $\nu = \frac{\nu_1}{\mu_1}$ ,  $\delta = \frac{\nu_2}{\mu_1}$ ,  $\beta = \frac{\beta_1}{\beta_2}$ , and our new  $x = x(\frac{\mu_1}{\beta_2})^{\frac{1}{2}}$ . Thus we have

$$\begin{cases} \frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} + ((N_1 - u) - \nu v)u \\ \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (\mu(N_2 - v) - \delta u)v. \end{cases} \quad (7)$$

We will now solve for the steady states of these equations. First, we will set  $u(x, t) = u(x - ct) = u(z)$  where  $z = x - ct$  and  $c$  is the wave speed. Thus,  $\frac{\partial u}{\partial t} = -c \frac{du}{dz}$  and  $\frac{\partial u}{\partial x} = \frac{du}{dz}$ . Similarly,  $v(x, t) = v(x - ct) = v(z)$ , so  $\frac{\partial v}{\partial t} = -c \frac{dv}{dz}$  and  $\frac{\partial v}{\partial x} = \frac{dv}{dz}$  [2]. Now, we let  $T = u'$  and  $W = v'$ . Thus we have

$$\begin{cases} T' = \frac{((N_1 - u) - \nu v)u + cT}{-\beta} \\ W' = (\mu(N_2 - v) - \delta u)v + cW \\ T = u' \\ W = v' \end{cases}$$

If we set these four equations equal to zero and represent the solution in the form  $(u, v, T, W)$ , we can easily see that three of the steady state solutions are  $(0, 0, 0, 0)$ ,  $(N_1, 0, 0, 0)$ , and  $(0, N_2, 0, 0)$ . We still need to find the fourth steady state. Because we set all equations equal to zero, we automatically know that  $W$  and  $T$  equal zero. Now we will assume that  $u$  and  $v$  are not equal to zero. We have

$$W' = 0 = (\mu(N_2 - v) - \delta u)v.$$

We solve for  $v$  and find that

$$v = \frac{\mu N_2 - \delta u}{\mu}.$$

We also know that

$$T' = 0 = \frac{((N_1 - u) - \nu v)u}{-\beta}.$$

We can disregard the  $-\beta$ , and when solving for  $u$ , we have

$$u = N_1 - \nu v.$$

If we substitute the expression we have for  $u$  into the equation we got for  $v$ , and solve for  $v$ , we get

$$v = \frac{\mu N_2 - \delta N_1}{\mu - \delta \nu}.$$

Thus,

$$u = N_1 - \nu \frac{\mu N_2 - \delta N_1}{\mu - \delta \nu}.$$

Hence, our final steady state is  $(N_1 - \nu \frac{\mu N_2 - \delta N_1}{\mu - \delta \nu}, \frac{\mu N_2 - \delta N_1}{\mu - \delta \nu}, 0, 0)$ .

We now must check to see if these steady states are stable or unstable. For each steady state, we need to use the following matrix of partials:

$$\begin{bmatrix} \frac{-2u + N_1 - \nu v}{-\beta} - \lambda & \frac{\nu u}{\beta} & \frac{c}{-\beta} & 0 \\ -\delta v & -2\mu v + \mu N_2 - \delta u - \lambda & 0 & c \\ 0 & 0 & 1 - \lambda & 0 \\ 0 & 0 & 0 & 1 - \lambda \end{bmatrix}$$

If we substitute  $(0, 0, 0, 0)$ , we see that the matrix becomes

$$\begin{bmatrix} \frac{N_1}{-\beta} - \lambda & 0 & \frac{c}{-\beta} & 0 \\ 0 & \mu N_2 - \lambda & 0 & c \\ 0 & 0 & 1 - \lambda & 0 \\ 0 & 0 & 0 & 1 - \lambda \end{bmatrix}$$

Solving the matrix for lambda, we see  $\lambda = 1, \frac{N_1}{-\beta}, \mu N_2$ . Thus, the trivial solution is unstable. We do the same for  $(N_1, 0, 0, 0)$  and find

$$\begin{bmatrix} \frac{N_1}{\beta} - \lambda & \frac{\nu N_1}{\beta} & \frac{c}{-\beta} & 0 \\ 0 & \mu N_2 - \delta N_1 - \lambda & 0 & c \\ 0 & 0 & 1 - \lambda & 0 \\ 0 & 0 & 0 & 1 - \lambda \end{bmatrix}$$

and  $\lambda = \frac{N_1}{\beta}, 1, \mu N_2 - \delta N_1$  and is again unstable. Similarly, for  $(0, N_2, 0, 0)$ , we have

$$\begin{bmatrix} \frac{N_1 - \nu N_2}{-\beta} - \lambda & 0 & \frac{c}{-\beta} & 0 \\ -\delta N_2 & -\mu N_2 - \lambda & 0 & c \\ 0 & 0 & 1 - \lambda & 0 \\ 0 & 0 & 0 & 1 - \lambda \end{bmatrix}$$



and  $\lambda = \frac{N_1 - \nu N_2}{-\beta}, 1, -\mu N_2$ , thus  $(0, N_2, 0, 0)$  is an unstable steady state. Finally, for  $(N_1 - \nu \frac{\mu N_2 - \delta N_1}{\mu - \delta \nu}, \frac{\mu N_2 - \delta N_1}{\mu - \delta \nu}, 0, 0)$ , we have

$$\begin{bmatrix} -N_1 + \nu \frac{\mu N_2 - \delta N_1}{\mu - \delta \nu} - \lambda & \nu \frac{N_1 - \nu \frac{\mu N_2 - \delta N_1}{\mu - \delta \nu}}{-\beta} & \frac{c}{-\beta} & 0 \\ -\delta \frac{\mu N_2 - \delta N_1}{\mu - \delta \nu} & -2\mu \frac{\mu N_2 - \delta N_1}{\mu - \delta \nu} + \mu N_2 - \delta N_1 + \delta \nu \frac{\mu N_2 - \delta N_1}{\mu - \delta \nu} - \lambda & 0 & c \\ 0 & 0 & 1 - \lambda & 0 \\ 0 & 0 & 0 & 1 - \lambda \end{bmatrix}$$

Note that  $\lambda = 1$  is a solution; therefore, this steady state is also unstable.

We have found that all four steady states of this “one dimensional” model are unstable. Now, it would be beneficial to have graphical representations of the population densities over space and time.

## 2.2 Computer Modeling of Two Species in a “One Dimensional” World

Knowing what the mathematical model is allows us to solve for steady states, but in order to understand how the species interact on their terrain, we must plot these equations. Using MATLAB, we were able to plot the population density of both species on the region  $[0, 1]$  on the real line throughout time. Remember that right now we are assuming that the land is flat, in other words, there is no height function.

We will first look at Species 1 and Species 2 and how they survive in the “one-dimensional” world. Recall Equation 6; in our first example, we set  $\mu_1 = 40$ ,  $\mu_2 = 35$ ,  $\nu_1 = 35$ ,  $\nu_2 = 17$ ,  $N_1 = 150$ , and  $N_2 = 100$ . Note that all of these figures pertain to the population density of the species.

We start with Species 1 having an initial population density of 48 and Species 2 having an initial population density of 53 on the whole terrain. We then see how the species will interact over time. Figure 1 shows the population density of Species 1 and Figure 2 shows the population density of Species 2. Note that Species 1 and Species 2 are in coexistence, even though the population density of Species 1 is larger than that of Species 2.

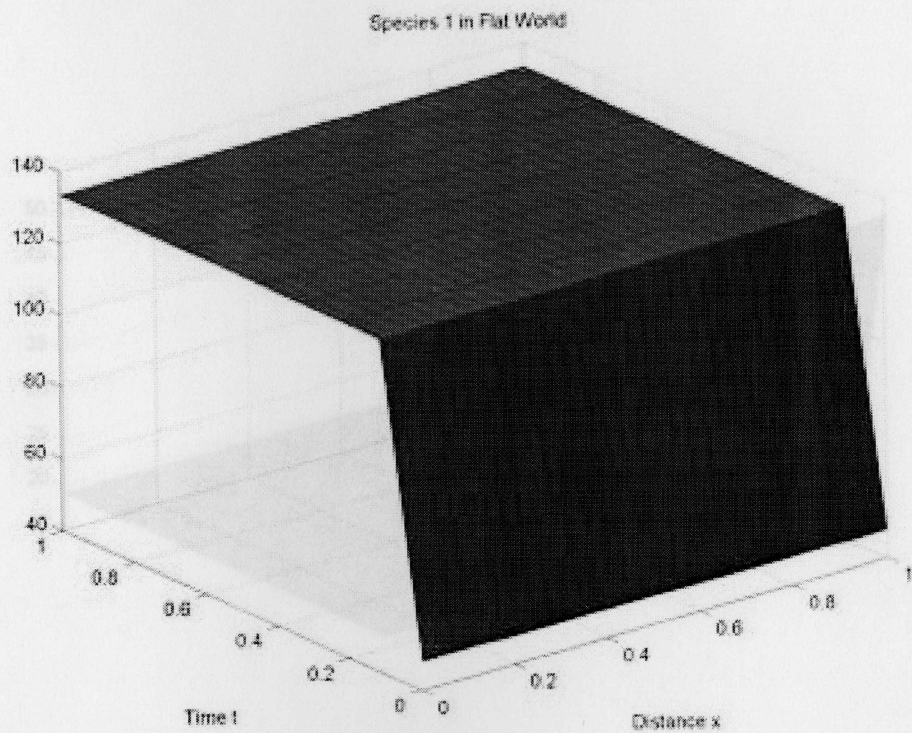


Figure 1

Recall that this model was unstable, thus we hope to see this from the computer program. We do this by changing one or more of the parameters slightly in order to see how it will transform the population densities of the two species. Hence, we kept everything the same except we made  $\mu_1 = 45$  and  $\nu_2 = 20$ , thus increasing the birth rate of Species 1 by five and the death rate of Species 2 by three. Now Species 1 reaches its carrying capacity whereas Species 2 becomes extinct (see Figures 3 and 4, respectively). This makes perfect sense since Species 2 already had a small population density compared to Species 1 and we increased its death rate.

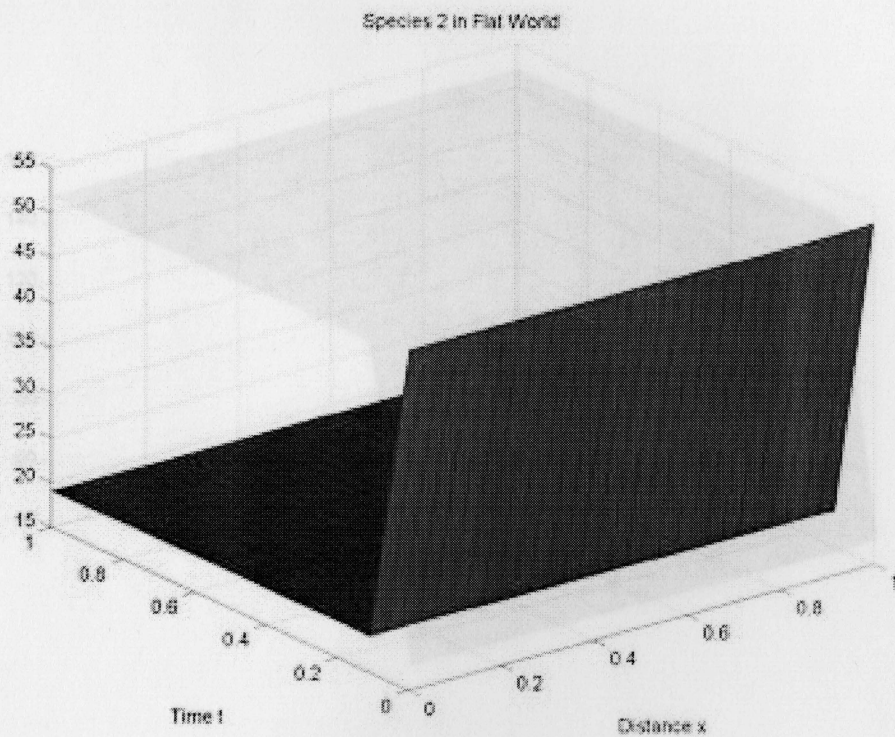


Figure 2

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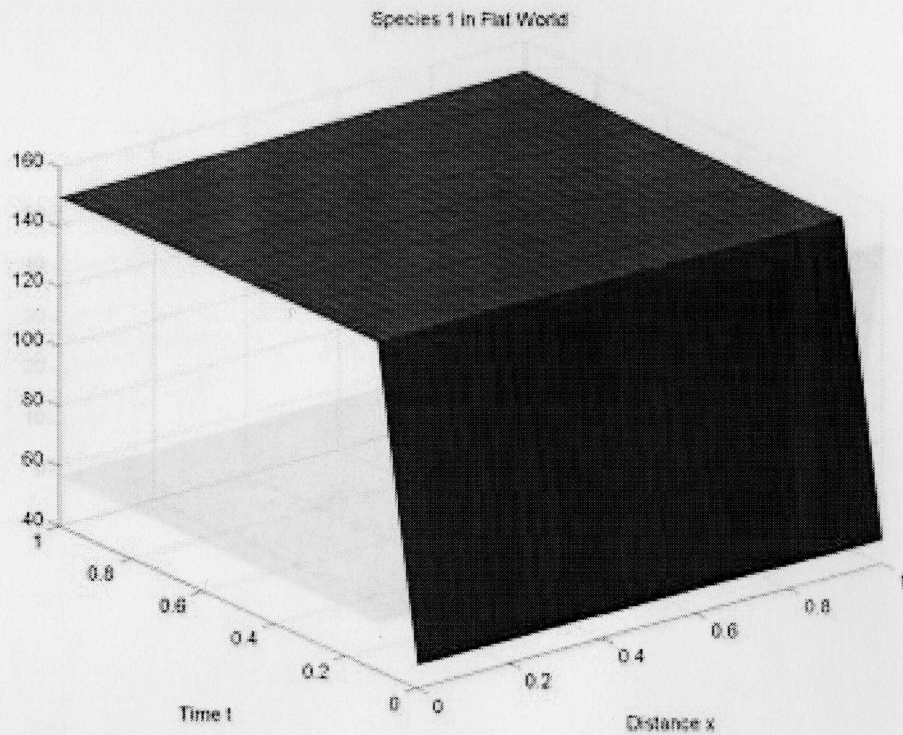


Figure 3

Therefore, we have seen that our model is exemplifying what we have mathematically shown: the values of the variables can drastically change the outcome of the species interaction due to the unstable nature of the model.

### 3.3 Modeling of Two Species in a "Two Dimensional" World

Now that we saw how the model would work in a "one dimensional" world, we will incorporate a height function into our model, allowing the species to live on a "two dimensional" world. Recall that the model we are using is the following:

$$\begin{cases} \frac{\partial \phi_1}{\partial t} - \Delta_x \phi_1 = (r_1(N_1 - \phi_1) - r_2 \phi_2) \phi_1 \\ \frac{\partial \phi_2}{\partial t} - \Delta_x \phi_2 = (r_2(N_2 - \phi_2) - r_1 \phi_1) \phi_2 \end{cases}$$

In order to incorporate a height function into our model, the height function needed to be both continuous and smooth on the terrain. For our model, we used the following height function:

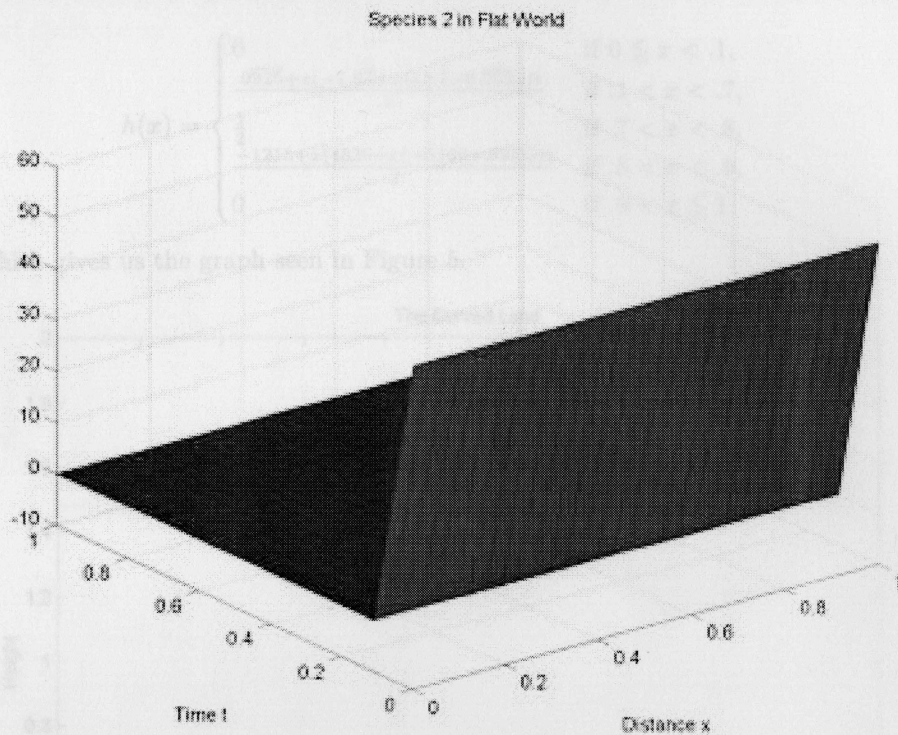


Figure 4

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$$\begin{cases} \frac{\partial \phi_1}{\partial t} - \beta_1 \Delta_s \phi_1 = (\mu_1(N_1 - \phi_1) - \nu_1 \phi_2) \phi_1 \\ \frac{\partial \phi_2}{\partial t} - \beta_2 \Delta_s \phi_2 = (\mu_2(N_2 - \phi_2) - \nu_2 \phi_1) \phi_2 \end{cases}$$

In order to incorporate a height function into our model, the height function needed to be both continuous and smooth on the terrain. For our model, we used the following height function:

prefers the higher elevations, which can also be seen through the different birth and death rates.

$$h(x) = \begin{cases} 0 & \text{if } 0 \leq x < .1, \\ \frac{.0925+x(-1.94+x(11.1-9.259x))}{2} & \text{if } .1 < x < .7, \\ \frac{1}{2} & \text{if } .7 < x < .8, \\ \frac{-1215+x(4320+x(-5100+2000x))}{2} & \text{if } .8 < x < .9, \\ 0 & \text{if } .9 < x \leq 1. \end{cases}$$

which gives us the graph seen in Figure 5.

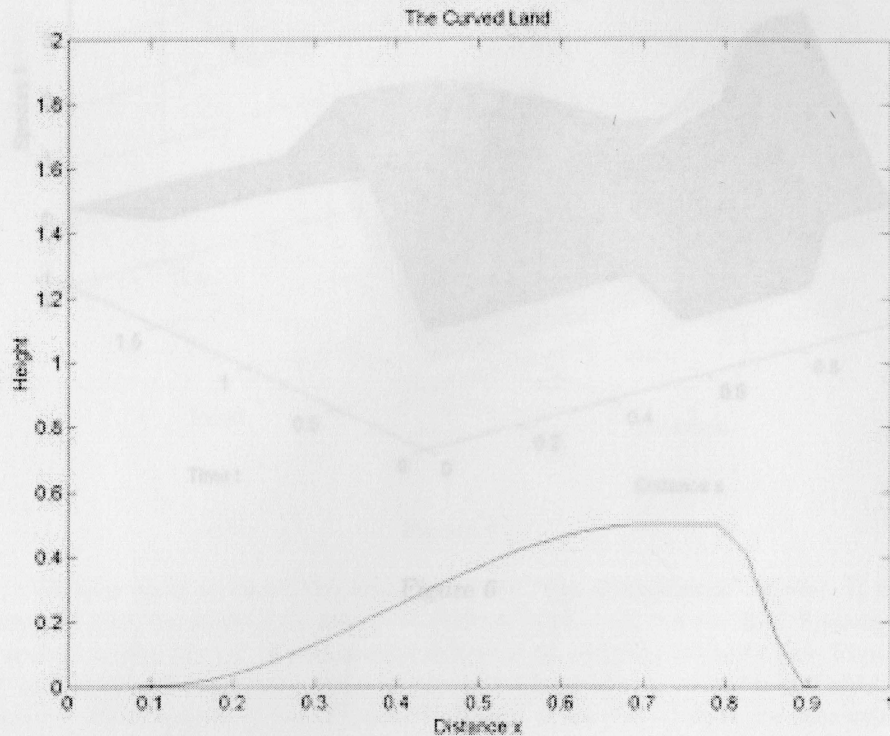


Figure 5

Now that we have the terrain that the species will live on, we can see how they will interact. We first will specify that Species 1 cannot live above a certain altitude; whereas, Species 2 can live above that altitude. Using the MATLAB program, we set  $\mu_1 = 0$ ,  $\mu_2 = 4$ ,  $\nu_1 = 3$ ,  $\nu_2 = 1$  for  $h(x) \geq .4$ , or  $.55 \lesssim x \lesssim .85$ , and  $\mu_1 = 3$ ,  $\mu_2 = 3$ ,  $\nu_1 = 2$ ,  $\nu_2 = 2$  elsewhere. We also have  $N_1 = N_2 = 10$  and the initial population density of each species is one on  $h(x) < .4$ . We see that Species 1 is approaching extinction whereas Species 2 survives on the whole terrain (see Figure 6 and 7, respectively). Also note that Species 2 reaches its carrying capacity on the region where  $h(x) > .4$ ; this shows that Species 2

prefers the higher elevation, which can also be seen through the different birth and death rates.

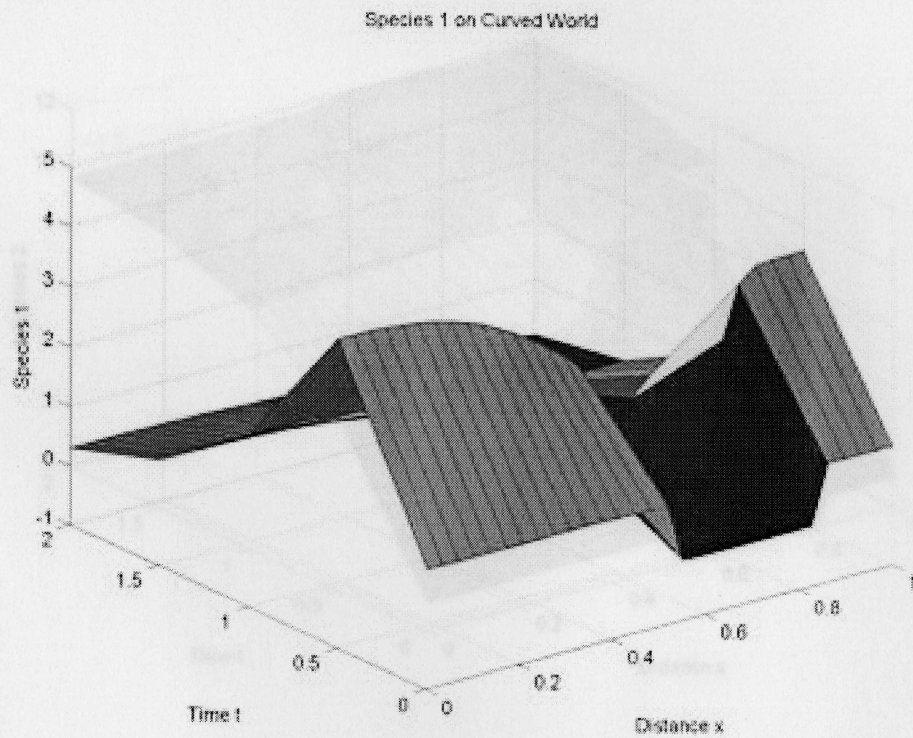


Figure 7

We also want to check the stability of a "two-dimensional" model. If we keep all else the same, but set  $\alpha = 1$ , when  $\lambda(x) < 4$ , we see that Species 1 survives where  $\lambda(x) < 4$  and that it achieves its carrying capacity (see Figure 5) and Species 2 reaches its carrying capacity where  $\lambda(x) > 4$  and is still able to survive when  $\lambda(x) < 4$  (see Figure 9). Hence, these two species are coexisting on the terrain. Thus, the terrain also appears to be unstable.

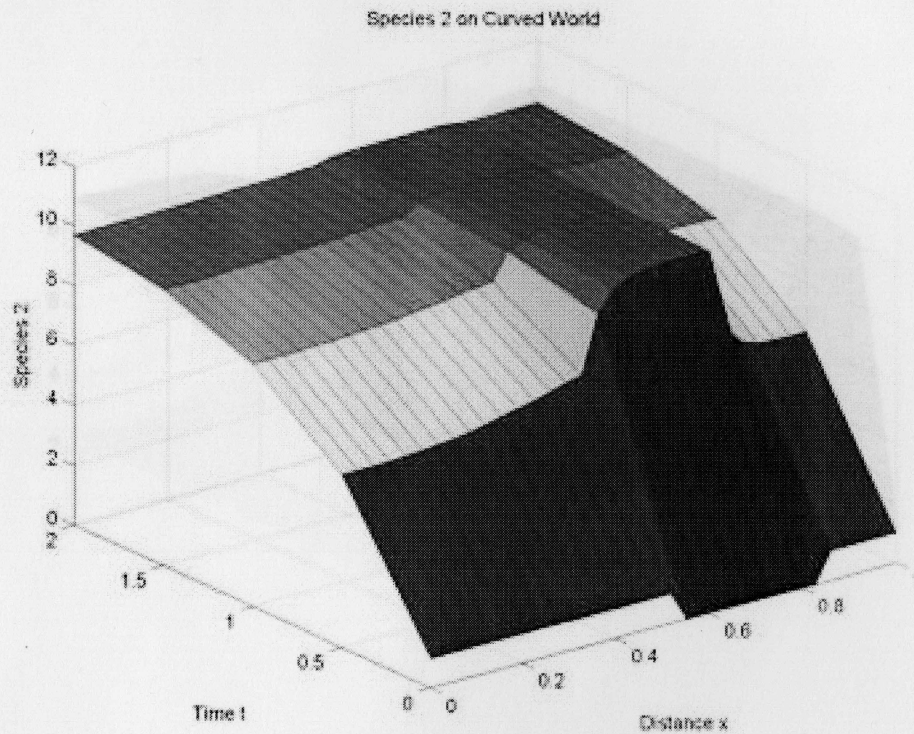


Figure 7

We also want to check the stability of the “two-dimensional” model. If we keep all else the same, but set  $\nu_1 = 1$  when  $h(x) < .4$ , we see that Species 1 survives where  $h(x) < .4$  and almost achieves its carrying capacity (see Figure 8) and Species 2 reaches its carrying capacity where  $h(x) > .4$  and is still able to survive when  $h(x) < .4$  (see Figure 9). Hence, these two species are coexisting on the terrain. Thus, this model also appears to be unstable.



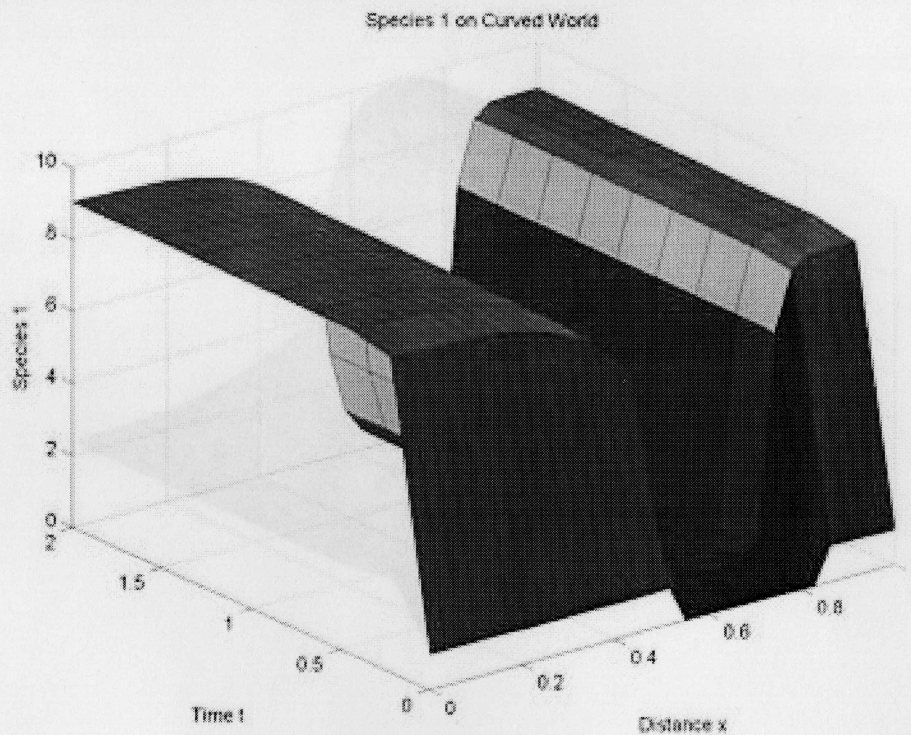


Figure 8

Therefore, this model gives us a more realistic outcome to how two species interact by including not only migration but also height.

### 3 CONCLUSION

We have shown that by adding both migration and space parameters into the population model, we have caused the model to be more realistic so that it parallels what we see happening in nature. Obviously, the addition of these parameters have also increased the complexity of the model causing us to rely on computer programming to plot the population density over time and space.

The obvious next step would be to make the birth and death rates functions of the terrain and see how that changes the model; the height function could also be altered to see how different terrains could lead to other outcomes given the same birth and death rates. Next, the boundary conditions could be altered, unfortunately, MATLAB does not yet have the capability to solve this model with boundary conditions other than Neumann. Other parameters could also be incorporated into the model; for example, the size of the species. Of course,

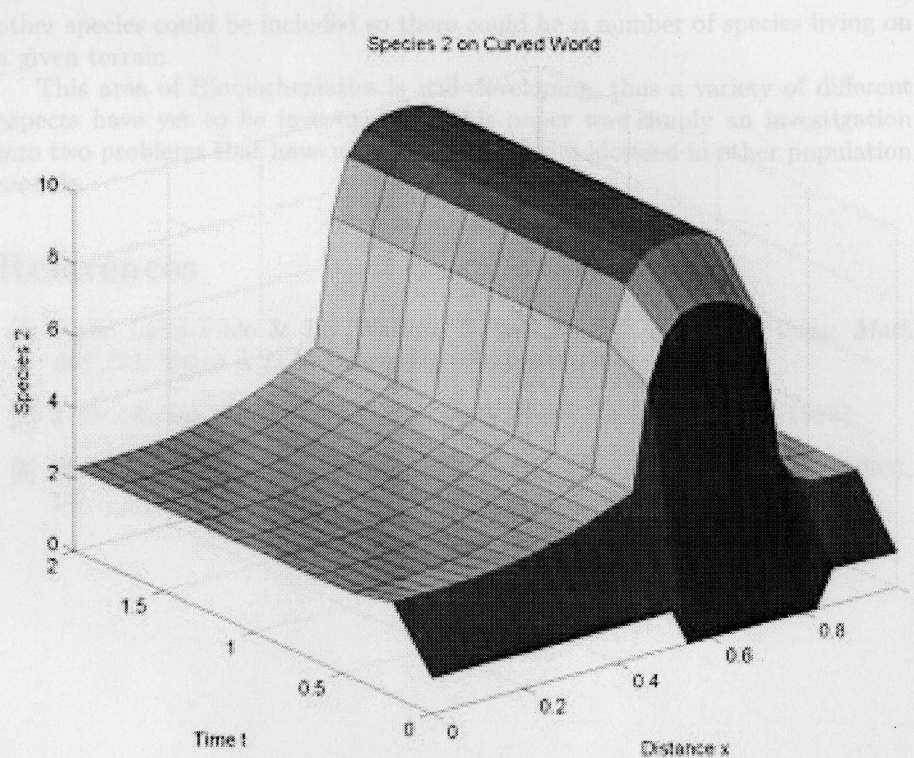


Figure 9

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### 3 CONCLUSION

We have shown that by adding both migration and space parameters into the population model, we have caused the model to be more realistic so that it parallels what we see happening in nature. Obviously, the addition of these parameters have also increased the complexity of the model causing us to rely on computer programming to plot the population density over time and space.

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other species could be included so there could be  $n$  number of species living on a given terrain.

This area of Biomathematics is still developing, thus a variety of different aspects have yet to be investigated. This paper was simply an investigation into two problems that have not been previously addressed in other population models.

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