The Shellability of Simplices

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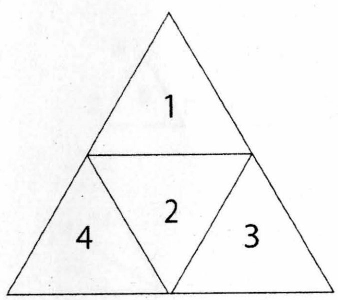
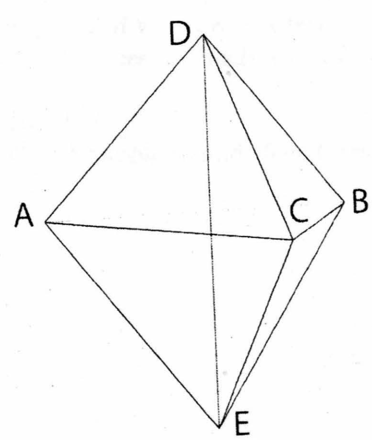
1 What is Shelling?

Shelling can be looked at as a way of piecing together a surface (or taking it apart), almost as if using legos. The pieces we are using are points, edges, triangles, tetrahedra, and larger n-dimensional cells. You can shell any triangulated surface. Let's begin with an example. In order to shell a triangulated disk, we begin by picking any triangle to shell. The only rule we must follow as we proceed is that the next triangle we shell must have a one dimensional intersection with the previously shelled part of the disk. Here is an example of a potential shelling of a triangulated disk.

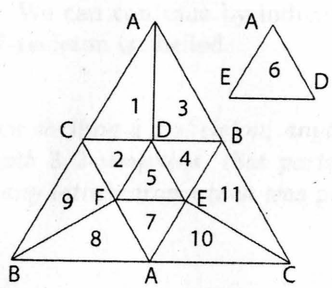


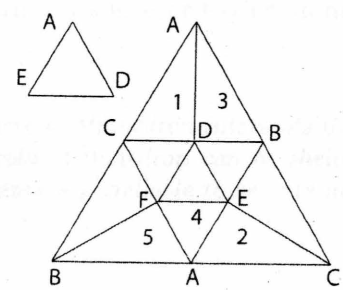
In this paper, we will only be shelling simplices. An n-simplex is an n-dimensional cell. For example, a 0-simplex is a point, a 1-simplex is an edge, a 2-simplex is a triangle, etc. In a simplex, every vertex is connected to every other vertex by an edge; every 3 edges bound a triangle; every 4 triangles bound a tetrahedron; etc. Here is an example of a possible shelling of a 4-simplex with vertices labeled A, B, C, D, E. In this case, we begin by choosing a tetrahedron to shell instead of a triangle, and when choosing which tetrahedron to shell next, the intersection between the next choice and the previously shelled simplices must be exactly a 2- dimensional instead of 1-dimesional.

Begin by shelling the tetrahedron $ABCD$
Next shell $ABCE$
Then $ABDE$
Then $ACDE$
Then $BCDE$



In general, when shelling an n-simplex β, you proceed by shelling one $(n-1)$-simplex at a time, and each $(n-1)$-simplex that you shell must have an $(n-2)$-dimensional intersection with the previously shelled portion of β.
You can also choose to shell the k-skeleton of an n-simplex. In this case, instead of choosing $(n-1)$-simplices to shell, you would choose k-simplices. Just like the examples above, when choosing which k-simplex to shell next, the intersection must be exactly $(k-1)$-dimensional.
What if a simplex is shellable, but only if you choose to shell its simplices in a certain order? Are there some simplices for which the order does not matter as long as you follow the rules? Yes! These simplices are called extendably shellable. For the remainder of this paper, we will be exploring which simplices are extendably shellable. Here is an example of something that is shellable, but not extendably shellable. If we shell it in the first order, we can complete the process, but if we choose the second order we get stuck.





2 Definitions and Notation

$\triangle^n$ denotes an n-simplex

A face of a simplex α is any simplex contained in α. Any face of α will have a lower dimension than α.

The star of a simplex α is the set of all simplices of which α is a face.

The closed star of a simplex is the set of all simplices of which α is a face and all of the faces of those simplices.

The link of a simplex is the closed star - the star.

The k-skeleton of a simplex α is the set of all k-simplices and their faces in α.

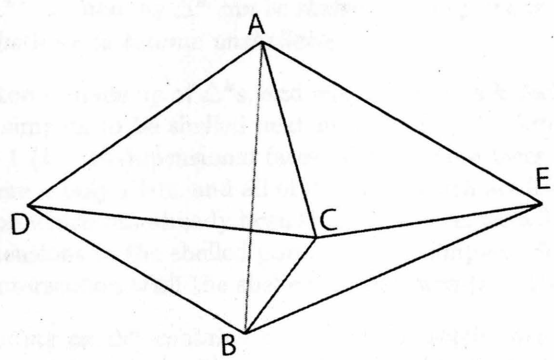
3 Proofs

Lemma 1. *The 2-skeleton of any n-simplex is extendably shellable.*

Proof. As we begin to shell the 2-skeleton of an n-simplex, ie the set of all triangles, edges, and vertices in the simplex, the first 2 triangles we choose to shell are arbitrary because all choices are equivalent. We begin with any triangle, and then choose any of the triangles which shares an edge with the first choice. At any point in the process, if a triangle, α, intersects the previously shelled portion of the simplex with 2 or 3 edges, then α can be shelled at any time without causing any other triangle to become unshellable. So without loss of generality, we can now shell the 2 triangles which, with our first 2 triangles, form a tetrahedron. Now we have another regular and symmetric form, so our next choice is an arbitrary one because all choices which intersect the tetrahedron are equivalent. After choosing a triangle to shell, we have caused 2 other triangles in our 2-skeleton to share 2 edges with the currently shelled portion, and as before, without loss of generality, we can shell those triangles now or at any point in our process. Whereas before we had a tetrahedron, we now have the 2-skeleton of a 4-simplex, another regular and symmetric form, which makes our next choice arbitrary. We can continue by induction, without ever having an opportunity to get stuck, until the entire 2-skeleton is shelled.

□

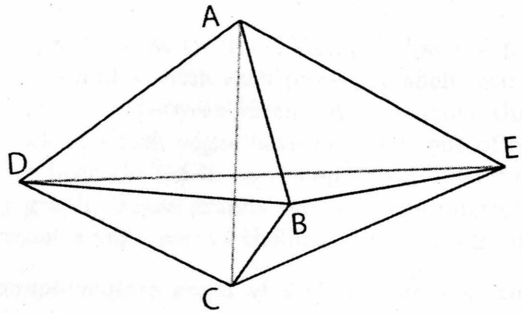
Lemma 2. *When shelling a 3-skeleton, anytime a tetrahedron intersects the previously shelled part of the simplex with 3 2-simplices, that particular tetrahedron can be shelled at any point in time without causing any tetrahedron which was previously shellable to become unshellable.*



Proof. If ABD, ACD, and BCD are already included in a partial shelling, then shelling ABCD would not cause any edges or vertices which were not already shelled to become shelled. Shelling an additional triangle without any additional edges or vertices can only cause more tetrahedra to become shellable. It cannot cause any tetrahedra which were shellable to become unshellable. In this figure, $ABCE$ is an example of a tetrahedron which would still be shellable after shelling $ABCD$ if it was already .

□

Lemma 3. *When shelling a 4-skeleton, anytime a $\triangle^4$ contains 3 shelled $\triangle^3$s, we can shell the other 2 $\triangle^3$s contained in that particular $\triangle^4$ at any time without causing any $\triangle^4$ which was previously shellable to become unshellable.*



Proof. If $ABDE$, $ACDE$, and $ABCD$ have already been shelled, then $BCDE$ and $ABCE$ can be shelled without causing any additional edges or vertices to be shelled. Shelling an additional triangle without any additional edges or vertices can only cause more tetrahedra to become shellable. It cannot cause any tetrahedra which were shellable to become unshellable. □

Lemma 4. *When shelling a k-skeleton, anytime a $\triangle^k$ intersects the previously shelled part of the simplex with k $\triangle^{k-1}$s, then the $\triangle^k$ can be shelled at any point in time without causing any $\triangle^k$ which was previously shellable to become unshellable.*

Proof. A k-skeleton is made up of $\triangle^k$s, and when shelling a k-skeleton, the intersection of the shelled portion and the simplex to be shelled next must be $(k-1)$-dimensional. If we are looking to shell a $\triangle^k$, it has $k+1$ $(k-1)$-dimensional faces. If k of those faces intersect the shelled portion of the simplex, then there is only 1 left, and all of its faces, which are less than $(k-1)$-dimensional, are also part of the portion which has already been shelled. Hence, we will not be adding any simplices of less than $(k-1)$ dimensions to the shelled portion of the simplex. So any simplex which was previously shellable, ie its intersection with the shelled portion was $(k-1)$-dimensional, is still shellable. □

Lemma 5. *Anytime an $\triangle^n$ contains $n-1$ $\triangle^{n-1}$s which have already been shelled, you can shell the other 2 $\triangle^{n-1}$s at any time without causing any $\triangle^{n-1}$ which was previously shellable to become unshellable.*

Proof. An $\triangle^n$ contains $n+1$ $\triangle^{n-1}$s, so if $n-1$ $\triangle^{n-1}$s have been shelled, there are 2 left. Each face of an $\triangle^{n-1}$ is an $\triangle^{n-2}$, and each $\triangle^{n-1}$ contains exactly n $\triangle^{n-2}$s. Each $\triangle^{n-1}$ in an $\triangle^n$ shares exactly 1 $\triangle^{n-2}$ with each other $\triangle^{n-1}$ in the simplex (for example, each tetrahedron in a $\triangle^4$ shares exactly 1 triangle with every other tetrahedron in the $\triangle^4$). Therefore, if there are 2 unshelled $\triangle^{n-1}$s, they share exactly 1 $\triangle^{n-2}$ with each other and exactly 1 $\triangle^{n-2}$ with every other $\triangle^{n-1}$ in the simplex, which have already been shelled by our assumption. So the intersection of each of our unshelled $\triangle^{n-1}$s with the shelled portion of the $\triangle^n$ is $n-1$ $\triangle^{n-2}$s, which is an intersection of dimenion $n-2$, and is therefore allowed, and each face of the $\triangle^{n-2}$ is also the face of a previously shelled $\triangle^{n-2}$ and was already shelled so that "filling in" the $\triangle^{n-2}$ does not cause anything to become unshellable. □

4 $(n-2)$-skeletons

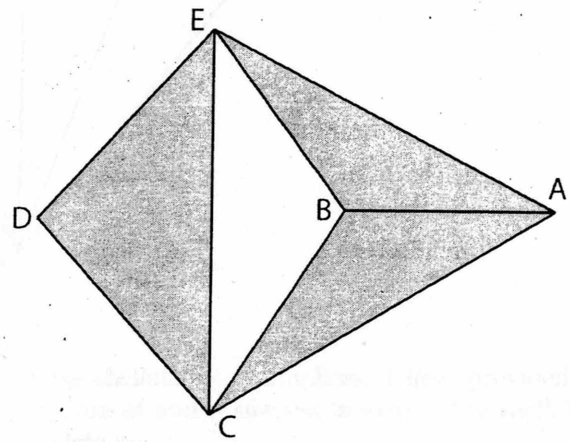
An n-simplex has $n+1$ vertices, so an $(n-2)$-simplex has $n-1$ vertices. Hence, when shelling the $(n-2)$-skeleton of an n-simplex, with each piece you shell, you are leaving out exactly 2 vertices of the n-simplex and the edge between them. As you move through the shelling process, it can be simpler to keep track of which edges have been left out of each step instead of keeping track of which simplices have been shelled in each step. I will refer to the graph formed by these edges as the complimentary graph. These graphs present some interesting questions. Does any graph of connected edges represent a valid partial shelling of the $(n-2)$-skeleton of an n-simplex?

Lemma 6. *In the complimentary graph of a partial shelling, any* 2 *edges must be connected by a path of* 1 *edge.*

Proof. Suppose we are shelling the $(n-2)$-skeleton of an n-simplex which consists of the vertices A, B, C, D, E, F, G,... Is it possible to have a partial shelling which includes the following:

ABC... (compliment DE)
ABE... (compliment CD)
CDE... (compliment AB)

Without having shelled the following:
ADE... (compliment BC)
BCE... (compliment AD)
ACD... (compliment BE)
ACE... (compliment BD)
BCD... (compliment AE)

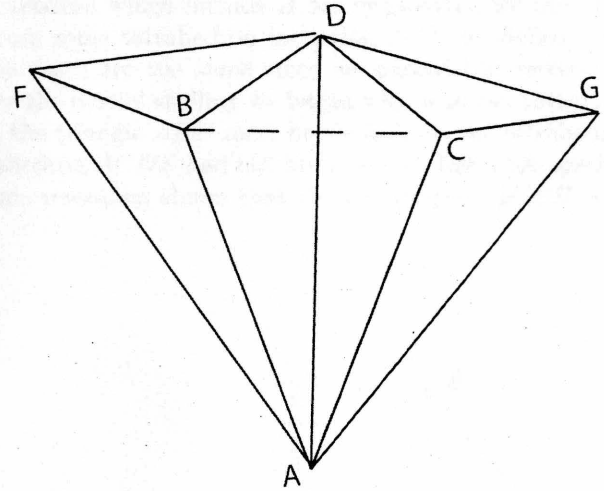


CDE... (compliment AB) cannot be shelled before ABC... (compliment DE) because the intersection is 0-dimensional. But conversely, ABC... (compliment DE) cannot be shelled before CDE... (compliment AB) for the same reason. Therefore, both cannot be included in a partial shelling unless something from list 2 is also included to make the intersection the right dimesion. □

5 3-skeletons

Lemma 7. *The closed star of any vertex in the* 3*-skeleton of an an n-simplex is extendably shellable.*

Proof. Consider the closed star, we'll call it α, of a vertex A in an n-simplex β. All tetrahedra in the closed star include the vertex A. Suppose not all vertices in α have been shelled. Then pick a shelled triangle which includes A and shell the unique tetrahedron which includes that triangle and the unshelled vertex. If all of the vertices in α have been shelled, then choose any 2 shelled triangles in the closed star which intersect each other with a line. These intersecting triangles must exist somewhere in the closed star because all vertices, and therefore edges including A, have been shelled, and each shelled tetrahedron intersects the rest of the partial shelling in exactly 2 dimensions. So, we have 2 shelled triangles whose intersection is a line. We'll call them ABD and ACD. Consider the triangle ABC. If the line BC is shelled, then the triangle ABC must be shelled because in the closed star, every tetrahedron includes A. So either the line BC is not shelled, or the triangle ABC is shelled. Either way, the tetrahedron $ABCD$ has a 2 dimensional intersection with the rest of the partial shelling. Therefore, we have a next move, and the closed star of a vertex in the 3-skeleton of an n-simplex is extendably shellable.



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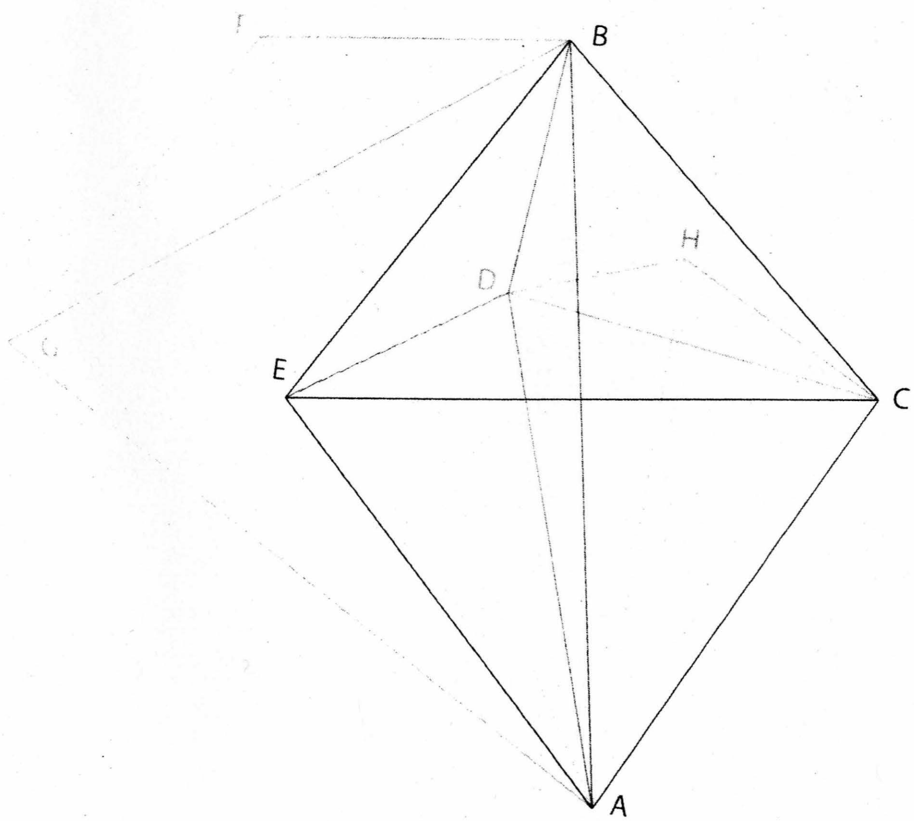
6 Further Questions

Here are some further questions about the extendable shellability of simplices. I have provided an extensive outline of a possible proof that the 3-skeleton of and n-simplex is extendably shellable. With a few added details, this could become a complete proof.

Proposition. The 3-skeleton of an n-simplex is extendably shellable.

Proof. The closed star of some vertex A includes every vertex in the simplex. It also includes all of the edges and all of the triangles in the simplex. However, it does not include any tetrahedra which do not include A, but as stated above, it does include all faces of those tetrahedra. The closed star is extendably shellable, but what if one or more of the tetrahedra not in the closed star were part of a partial shelling? Is there still a next move?

Suppose we begin with a partial shelling in the closed star of A. Then, suppose a tetrahedron which is not in the closed star, $BCDE$, is shelled. $BCDE$ shares each of its 4 triangles with a tetrahedron that is in the star. If all 4 of those tetrahedra are shelled, then there is essentially no difference in the partial shelling than if $BCDE$ weren't shelled, so there is a next move because the closed star is extendably shellable. Suppose not all 4 of those tetrahedra are shelled. At least 1 tetrahedron must be shelled because $BCDE$ must have a 2-dimensional intersection with the partial shelling, which by hypothesis includes some tetrahedron in the closed star. So now we have at least 2 shelled tetrahedra, one in the closed star, $ABCD$, and $BCDE$. Consider the line AB If this line is not shelled, then we can shell the tetrahedron $ABCD$, so there is a next move. If this line is shelled, then it is part of a shelled tetrahedron. This tetrahedron could be in the $\triangle^4$ $ABCDE$, in which case there is a next move because you could shell either of the other 2 tetrahedra in $ABCDE$. So suppose there is some other shelled tetrahedron, call it $ABFG$ for generality. In order for the partial shelling we began with, which was in the closed star of A, to include both $ACDE$ and $ABFG$, there must be other shelled tetrahedra which connect the two through a series of 2 dimensional intersections. So there must be at least one shelled tetrahedron which shares a triangle with $ACDE$. Since the 3 triangles in this tetrahedron which include A are symmetric, we can choose one without loss of generality. Let's suppose some tetrahedron including ACD is shelled. This could be $ACDF$ or $ACDG$. Both of these cases are the same since we named the vertices arbitrarily. So Suppose $ACDF$ is shelled. Since the partial shelling we began with is in the closed star of A, we know that if the line EF is shelled, the triangle AEF must be shelled. So the tetrahedron $ACEF$ is a potential next move. If the tetrahedron $ACDF$ had not been shelled, but a tetrahedron including some other point H, the exact same reasoning shows that the tetrahedron $ACEH$ would be a potential next move.



If the entire closed star were to be shelled before anything outside the closed star, the simplex would still be extendably shellable because once the closed star is shelled, all of the triangles are shelled and you're essentially filling in the tetrahedra.

$\square$

It has yet to be proven that the k-skeleton of an n-simplex is extendably shellable. Perhaps a proof that the 3-skeleton of an n-simplex is extendably shellable could be generalized to include all dimensions.